Ученые записки Таврического национального университета
им. В. И. Вернадского

Серия «Физико-математические науки»
Том 27 (66) № 1 (2014), с. 261–279.

УДК 519.833.2 MSC2000: 91A06

V. I. Zhukovskiy, S. N. Sachkov, L. V. Smirnova

EXISTENCE OF BERGE EQUILIBRIUM IN MIXED STRATEGIES

We formalize a guaranteed solution notion for a non-cooperative game of n persons under uncertainty. This notion is based on the appropriate modification of maximin and the Berge-Vaisman equilibrium. We obtain existence conditions for the guaranteed solution in the class of mixed strategies (probability measures).[1]

Keywords: probability measure, mixed strategy, weak compactness in itself, guarantee, Berge-Vaisman equilibrium, Nash equilibrium, maximin.

E-mail: zhkvlad@yandex.ru

Introduction

The Berge equilibrium concept was introduced *intuitively* by French mathematician Claude Berge [1]. A brief review of Berge's book by Shubik [2] scared economists and contributed to its subsequent neglect in the English-speaking world. Particularly it was marked: "The arguments have been presented in a rather abstract manner and no attention has been paid to applications to economics. The book will be of a little direct interest to economists". The Berge's book was translated into Russian in 1961, and V.Zhukovskiy in 1994-1995 formalized the Berge equilibrium for linear-quadratic differential games under uncertainties [3], [4].

Note that Nash equilibrium is a common optimality concept for non-cooperative games. The key difference is that in case of Nash equilibrium an individual player's deviation from the equilibrium cannot increase the player's own payoff whereas; at the same time in case of Berge equilibrium a deviation by one or more players can reduce the payoff of a player, who does not deviate from an equilibrium situation. The Berge

[1]Работа выполнена при поддержке РФФИ, проект №14-01-90408 Укр_а.

equilibrium concept formalizes mutual support among players motivated by the altruistic social value orientation in such games.

Now turn to formal definitions. Consider a non-cooperative game of three persons

$$\Gamma_3 = \langle \{1,2,3\}, \{X_i\}_{i=1,2,3}, \{f_i(x)\}_{i=1,2,3} \rangle ,$$

where $X_i \subset \mathbb{R}^{l_i}$ is a set of strategies x_i of the i-th player, $f_i(x)$ is his payoff function, a situation $x = (x_1, x_2, x_3) \in X = \prod_{i=1}^{3} X_i$.

K. Vaisman called a couple $(x^{\mathrm{v}}, f^{\mathrm{v}}) \in X \times \mathbb{R}^3$ a *Berge equilibrium solution for the game* Γ_3 [5]-[7] if the following conditions

(1^0) a situation $x^{\mathrm{v}} = (x_1^{\mathrm{v}}, x_2^{\mathrm{v}}, x_3^{\mathrm{v}})$ satisfies a *Berge equilibrium condition*, i.e.

$$f_1(x_1^{\mathrm{v}}, x_2, x_3) \le f_1(x^{\mathrm{v}}) \ \ \forall\, x_j \in X_j \ (j = 2,3),$$

$$f_2(x_1, x_2^{\mathrm{v}}, x_3) \le f_2(x^{\mathrm{v}}) \ \ \forall\, x_k \in X_k \ (k = 1,3),$$

$$f_3(x_1, x_2, x_3^{\mathrm{v}}) \le f_3(x^{\mathrm{v}}) \ \ \forall\, x_r \in X_r \ (r = 1,2);$$

(2^0) a property of *individual rationality holds* for all players, i.e.

$$f_1(x^{\mathrm{v}}) \ge \max_{x_1} \min_{x_2,x_3} f_1(x),$$

$$f_2(x^{\mathrm{v}}) \ge \max_{x_2} \min_{x_1,x_3} f_2(x),$$

$$f_3(x^{\mathrm{v}}) \ge \max_{x_3} \min_{x_1,x_2} f_3(x)$$

hold.

The "game"sense of the condition (2^0) is as follows. If the property of individual rationality (2^0) holds then every player provides himself a payoff $f_i(x^{\mathrm{v}})$ $(i = 1,2,3)$ which is at least not less than the i-th player's maximin. The condition (2^0) first was proposed by Zhukovskiy's doctoral student Konstantin Vaisman in 1994 [5]-[7]. He constructed some examples such that a situation satisfying the Berge equilibrium conditions (1^0) provides some players payoffs which are less than their maximins. In order to overcome this negative property of the Berge equilibrium Vaisman proposed to use additionally condition (2^0). Moreover, the following results were obtained by Vaisman:

– in some cases the Berge equilibrium exists, when there is no Nash equilibrium;
– in some games (The Prisoners' Dilemma, The Environmental Protection [8, p. 193]) if players simultaneously choose Berge equilibrium strategies, then everyone receives a larger payoff than if they chose Nash equilibrium strategies.

Konstantin Vaisman died suddenly at the age of 35 in 1998. He owned a remarkable trait: he had been helping everyone and forgetting himself. The authors of this paper think that Vaisman's researches of Berge equilibrium provide a basis to call the above mentioned solution $(x^{\mathrm{v}}, f^{\mathrm{v}})$ of the non-cooperative game Γ_3 the *Berge-Vaisman equilibrium*.

Sufficient existence conditions of the Berge equilibrium were obtained by Zhukovskiy [9] in the form of existence conditions for a saddle point $(x^0, z^{\mathrm{v}}) \in X \times X$ of the Germayer convolution $\max_i (f_i(x\|z_i) - f_i[z])$, $z_i \in X_i$, $z = (z_1, \dots, z_n) \in X = \prod_{i \in N} X_i$. The ideas of this approach have been used in the current research.

The aim of this paper is

– to formalize the Berge guaranteed solution for the non-cooperative game of n persons under uncertainty, when we have only the limits of variations of these uncertainties;
– to prove existence of the Berge guaranteed solution in the class of mixed strategies (probability measures).

1. Auxiliary data

1.1. **Existence conditions for continuous selector.** First introduce some facts from mathematical programming [10], [11].

Suppose that

(1) a set $X \subset \mathbb{R}^l$ ($\mathbb{R}^l$ is the Euclidean l-dimensional space) is a compact one;
(2) a set $Y \subset \mathbb{R}^m$ is a convex compact one;
(3) a scalar function $F(x, y)$ is determined and continuous on $X \times Y$, $x \in X$ and $y \in Y$;
(4) for any $x \in X$ the function $F(x, y)$ is strictly convex in $y \in Y$, i.e.

$$F(x, \lambda y^{(1)} + (1 - \lambda) y^{(2)}) < \lambda F(x, y^{(1)}) + (1 - \lambda) F(x, y^{(2)})$$

for all $y^{(j)} \in Y$ $(j = 1, 2)$ and any $\lambda = \mathit{const} \in (0, 1)$.

Then there exists a continuous m-vector function $y(x) : X \to Y$ such that

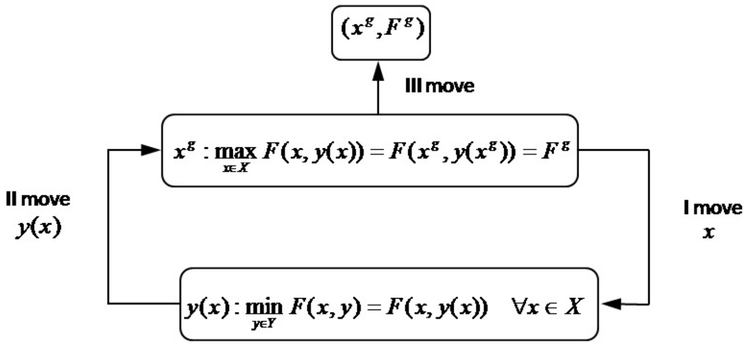
$$\min_{y \in Y} F(x, y) = F(x, y(x)) \ \ \forall x \in X.$$

1.2. **Maximin in terms of hierarchical game.** In game theory a maximin strategy x^g and a maximin F^g are defined by the chain of equalities

$$\max_{x \in X} \min_{y \in Y} F(x, y) = \min_{y \in Y} F(x^g, y) = F^g. \tag{1}$$

For the process of accepting a guaranteed solution $(x^g, F^g) \in X \times \mathbb{R}$ we can suggest the following interpretation in terms of bilevel hierarchical game [10]. Two players are participating in the game: Center and a player at a lower level on the hierarchy. Assume that Center forms his own strategy $x \in X$ and the player at the lower level constructs an uncertainty $y(x) : X \to Y$, $y(\cdot) \in C(X, Y)$ (see Fig. 1). The game proceeds as follows.

The first move is made by Center. He informs the lower level player of his possible strategies $x \in X$.



Pic. 1

The following (second) move is transferred to the lower level player, who forms an uncertainty $y(x) : X \to Y$ such that for every $x \in X$

$$\min_{y \in Y} F(x, y) = F(x, y(x)) = F[x], \tag{2}$$

and informs Center about a specific type of the uncertainty $y(x)$.

Finally (third move): Center forms a pair (x^g, F^g) which is defined by the condition

$$\max_{x \in X} F(x, y(x)) = F(x^g, y(x^g)) = F^g. \tag{3}$$

Thus, Center can use the strategy x^g. In this case Center provides himself the guarantee F^g whatever uncertainty $y(x) \in Y$ has been realized because $F^g \leq F(x^g, y) \ \forall y \in Y$. Since $F^g \geq F[x] \ \forall x \in X$, the guarantee F^g is the largest of all guaranties $F[x]$.

The given above "hierarchical"approach will be applied in Section 2.

1.3. **Mathematical model of conflict.** Assume that a mathematical model of conflict is represented by a non-cooperative game of N persons under uncertainty

$$\Gamma_N = \langle N, \{X_i\}_{i \in N}, Y^X, \{f_i(x, y)\}_{i \in N} \rangle.$$

Here $N = \{1, \ldots, n\}$ is a set of the agents (players) numbers. A strategy of the i-th player $x_i \in X_i \subset \mathbb{R}^{l_i}$ $(i \in N)$. The players choose their strategies independently of each other in the game Γ_N. Each i-th player formes and uses his own strategy $x_i \in X_i$ $(i \in N)$. As a result we get a situation $x = (x_1, \ldots, x_n) \in X = \prod_{i \in N} X_i \subset \mathbb{R}^l$ $(l = \sum_{i \in N} l_i)$. A set of uncertainties $y(x) : X \to Y \subset \mathbb{R}^m$ is denoted as Y^X. In the terminology of the theory of zero-sum games $y(x)$ is a *countersituation*. We define a payoff function $f_i(x, y)$ of the i-th player on the sets $(x, y(x))$. The i-th player obtains the payoff $f_i(x, y(x))$ which is equal to the value of his payoff function in the concrete couple $(x, y(x))$. The aim of the i-th player is to choose a strategy $x_i \in X_i$ such that his payoff is rational according to his point of view. By choosing their strategies the players need to focus on possibility of realization of any uncertainty $y(x) \in Y^X$.

Let us now turn to the notion of a guaranteed solution of the game Γ_N.

2. Guaranteed solution of game Γ_N

2.1. **Definition.** To formalize a solution of the game Γ_N we shall use the approach from Subsection 1.2. The only difference is that we replace formation of the *interior minimum* from (2) by formation of n minimums (for every i-th player)

$$f_i[x] = \min_{y \in Y} f_i(x, y) = f_i(x, y^{(i)}(x)) \ \ \forall i \in N, \ \ x \in X. \tag{4}$$

Moreover we replace formation of the *outer maximum* from (3) by two following operations:

a) find a set X^{v} of all situations x^{v} in the "game of guaranties"

$$\Gamma_g = \langle N, \{X_i\}_{i \in N}, \{f_i[x] = f_i((x, y^{(i)}(x))\}_{i \in N} \rangle$$

such that the *Berge equilibrium condition* is satisfied, i.e.

$$\max_{x_{N \setminus \{i\}} \in X_{N \setminus \{i\}}} f_i[x \| x_i^{\mathrm{v}}] = f_i[x^{\mathrm{v}}] \ \ (i \in N), \tag{5}$$

where

$$[x \| x_i^{\mathrm{v}}] = [x_1, \ldots, x_{i-1}, x_i^{\mathrm{v}}, x_{i+1}, \ldots, x_n], \ \ x_{N \setminus i} = [x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n],$$

$$X_{N \setminus \{i\}} = \prod_{j \in N, j \neq i} X_j;$$

b) find a Slater maximal situation $\bar{x}^{\mathrm{v}} \in X^{\mathrm{v}}$ in the n-criteria problem

$$\langle X^{\mathrm{v}}, \{f_i[x]\}_{i \in N} \rangle \tag{6}$$

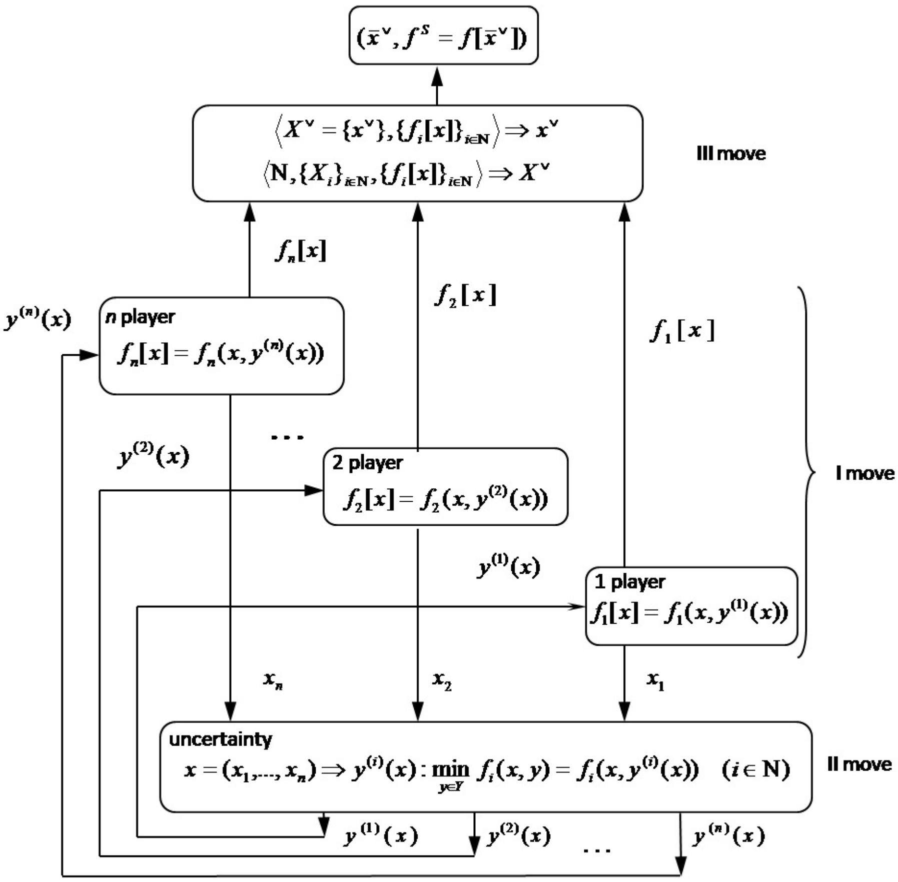
such that the system of strict inequalities

$$f_i[x] > f_i[\bar{x}^{\mathrm{v}}] = f_i^s \ \ (i \in N) \tag{7}$$

is inconsistent for any $x \in X^{\mathrm{v}}$.

Then the couple $(\bar{x}^{\mathrm{v}}, f^s) \in X \times \mathbb{R}^n$ is called a *Berge strong guaranteed solution (equilibrium) in the game* Γ_N. Here the n-vector $f = (f_1, \ldots, f_n) \in \mathbb{R}^n$. We present a construction process of the Berge strong guaranteed solution in the game Γ_N in Fig. 2.

Now introduce a formal definition.



Pic. 2

Definition 1. A couple $(\bar{x}^{\mathrm{v}}, f^s) \in X \times \mathbb{R}^n$ in the problem Γ_N is called a *Berge strong guaranteed solution (BVSGS)* if there exist n continuous m-vector functions $y^{(i)}(x)$ $(i \in$

$\in N$) such that

$$\min_{y(\cdot)\in Y^X} f_i(x, y(x)) = f_i(x, y^{(i)}(x)) = f_i[x] \ \ \forall x \in X \ \ (i \in N),$$

and

1^0) there exists a situation $x^{\mathrm{v}} \in X$ in the non-cooperative "game of guarantees"

$$\langle N, \{X_i\}_{i\in N}, \{f_i[x]\}_{i\in N}\rangle \tag{8}$$

such that the equality (5) is satisfied. The set of all situations x^{v} is designated by X^{v} ;

2^0) the situation $\bar{x}^{\mathrm{v}}$ is Slater maximal for the n-criteria problem

$$\langle X^{\mathrm{v}}, \{f_i[x]\}_{i\in N}\rangle,$$

i.e. for any $x \in X^{\mathrm{v}}$ there exists an index $j(x) = j \in N$ such that

$$f_j(x) \leq f_j(\bar{x}^{\mathrm{v}}) = f_j^s.$$

This condition is equivalent to inconsistency of the system $f_i[x] > f_i[\bar{x}^{\mathrm{v}}] = f_i^s$ $(i \in N)$ for any $x \in X^{\mathrm{v}}$.

3^0) the n-vector $f^S = (f_1[\bar{x}^{\mathrm{v}}], \ldots, f_n[\bar{x}^{\mathrm{v}}]) = (f_1^S, \ldots, f_n^S)$.

2.2. **Sufficient conditions for the Berge equilibrium.** We assign the Germayer convolution [12]

$$\varphi(x, z) = \max_{i\in N}(f_i[x\|z_i] - f_i[z]) \tag{9}$$

to the "game of guarantees"(8). Here

$$[x\|z_i] = [x_1, \ldots, x_{i-1}, z_i, x_{i+1}, \ldots, x_n] \in X = \prod_{i\in N} X_i, \ \ z_i \in X_i \ \ (i \in N),$$

$$z = [z_1, \ldots, z_i, \ldots, z_n] \in X.$$

A saddle point (x^0, z^v) *of the scalar function* $\varphi(x, z)$ is determined by the chain of inequalities

$$\varphi(x, z^{\mathrm{v}}) \leq \varphi(x^0, z^{\mathrm{v}}) \leq \varphi(x^0, z) \ \ \forall x, z \in X. \tag{10}$$

Taking into account (9) from the left inequality in (10) for $z^{\mathrm{v}} = x^0$ we get

$$\varphi(x^0, x^0) = \max_{i\in N}(f_i[x^0\|x_i^0] - f_i[x^0]) = 0.$$

Then (10) yields

$$\varphi(x, z^{\mathrm{v}}) = \max_{i\in N}(f_i[x\|z_i^{\mathrm{v}}] - f_i[z^{\mathrm{v}}]) \leq 0 \ \ \forall x \in X.$$

Hence for every $i \in N$

$$f_i[x\|z_i^{\mathrm{v}}] - f_i[z^{\mathrm{v}}]) \leq 0 \ \ \forall x \in X.$$

Thus, we get for all $x \in X$

$$f_i[x\|z_i^{\mathrm{v}}] \leq f_i[z^{\mathrm{v}}]) \ \ (i \in N). \tag{11}$$

Fulfillment of the conditions (11) for all $x \in X$ $(i \in N)$ means that the second component $z^{\mathrm{v}} = x^{\mathrm{v}} \in X$ of the saddle point (x^0, z^{v}) satisfies the Berge equilibrium condition (5).

Remark. Construction of the Berge equilibrium situation $z^{\mathrm{v}} = x^{\mathrm{v}} \in X$ (which is determined by (5)) is reduced to construction of the saddle point $(x^0, z^{\mathrm{v}}) \in X^2$ of the scalar function (9). The second component $z^{\mathrm{v}} \in X$ of the saddle point (x^0, z^{v}) satisfies the Berge equilibrium condition (5).

2.3. **Continuity of function $\varphi(x, z)$.**

Proposition. Suppose that in the "game of guarantees"(8) the following conditions

1. the sets X_i are compact ones (i.e. closed and bounded) in $\mathbb{R}^{l_i}$;
2. the payoff functions $f_i[x]$ are continuous on X $(i \in N)$

take place. Then the scalar function $\varphi(x, z)$ from (9) is continuous on $X \times X$.

This proposition follows immediately from the well-known property [11]: suppose that the function $\psi(u, w)$ is continuous on $U \times W$ and the set W is compact; then the function $\eta(u) = \max\limits_{w \in W} \psi(u, w)$ is continuous on U.

3. Mixed strategies

3.1. **Borel σ-algebra.** We consider the segment $Y^* = [y_1, y_2] \subset \mathbb{R}$. A collection $\Im$ of subsets of $Y = \{y \in \mathbb{R} | y_1 \leq y \leq y_2\}$ is called σ-algebra if it satisfies the following three properties:

1. $[y_1, y_2]$ is an element of $\Im$;
2. if $T \subset [y_1, y_2]$ is an element of $\Im$, then its complement $[y_1, y_2] \setminus T$ is an element of $\Im$ as well;
3. if $T_k \subset [y_1, y_2]$ $(k = 1, 2, \dots)$ are an elements of $\Im$ then their union $\bigcup\limits_{k=1}^{\infty} T_k$ is an element of $\Im$ too.

If every element of σ-algebra $\Im^{(1)}$ is an element of σ-algebra $\Im^{(2)}$ then one can say that *σ-algebra $\Im^{(2)}$ contains σ-algebra $\Im^{(1)}$*.

First, we consider any σ-algebra $\Im$ which contains all segments $[\alpha, \beta] \subset [y_1, y_2]$. One can prove that there exists a smallest σ-algebra $B(Y^*)$ such that

1. it is an element of any other σ-algebra;
2. all closed segments from $[y_1, y_2]$ are the elements of $B(Y^*)$.

This σ-algebra $B(Y^*)$ is called the *Borel σ-algebra*. Elements of the Borel σ-algebra $B(Y^*)$ are called *Borel measurable sets*. Thus for the segment $[y_1, y_2]$ the Borel σ-algebra is the smallest σ-algebra over $[y_1, y_2]$ containing all closed subsets of $[y_1, y_2]$.

Second, for the set $Y^* = \{y = (y_1, \dots, y_m) \mid y_i \in [y_i^{(1)}, y_i^{(2)}]\ (i = 1, \dots, m)\}$ σ-algebra $\Im$ is a collection of subsets of Y^* such that

1. Y^* is an element of $\Im$;
2. $\Im$ is closed with respect to the complementation operation $Y^*\backslash Y_k$ for all $Y_k \in \Im$ $(k = 1, 2, \ldots)$;
3. $\Im$ is closed with respect to the operation of countable unions $\bigcup\limits_{k=1}^{\infty} Y_k$.

The Borel σ-algebra $B(Y^*)$ is the smallest σ-algebra over Y^* containing all closed subsets of Y^*.

Third, we consider a set $Y \in \mathbb{R}^m$. Let Y be compact and hence bounded. Then there exist numbers $y_i^{(1)}$, $y_i^{(2)}$ $(i = 1, \ldots, m)$ such that

$$Y \subset Y^* = \{y = (y_1, \ldots, y_m) \mid y_i^{(1)} \le y \le y_i^{(2)}\ (i = 1, \ldots, m)\}.$$

Let us construct $B(Y^*)$. Then

$$B(Y) = B(Y^*) \bigcap Y = \{Y_k \bigcap Y \mid Y_k \in B(Y^*)\}.$$

The Borel σ-algebra $B(X_i)$, where the set X_i $(i \in N)$ of pure strategies x_i of the i-th player is a compact set in $\mathbb{R}^{l_i}$, is constructed in the same way.

3.2. **Mixed strategies and situations in mixed strategies.** Assume that in the class of pure strategies $x_i \in X_i$ $(i \in N)$ there does not exist a situation x^{v} satisfying the Berge equilibrium condition (5). Then one can follow the approach proposed by Borel, von Neumann and Nash. The approach is that the set X_i of pure strategies x_i should be extended to the set of mixed strategies; then for the game (8)

$$\langle N, \{X_i\}_{i \in N}, \{f_i[x]\}_{i \in N} \rangle$$

existence of situation satisfying the Berge equilibrium condition can be established on a class of mixed strategies.

For this construct the Borel σ-algebra $B(X_i)$ for every set X_i $(i \in N)$ and construct the Borel σ-algebra $B(X)$ for the set of situations $X = \prod\limits_{i \in N} X_i$. Assume that $B(X)$ contains all Cartesian products of elements of the Borel σ-algebras $B(X_i)$ $(i \in N)$.

In game theory a mixed strategy $\nu_i(\cdot)$ of the i-th player is a probability measure on the compact set X_i. Consider the definition from [8]. Assume that $B(X_i)$ is a Borel σ-algebra over a compact set $X_i \subset \mathbb{R}^{l_i}$. A probability measure is a nonnegative scalar function $\nu_i(\cdot)$ which is defined on $B(X_i)$ and satisfies the following two conditions:

1^0) for every sequence $\{Q_k^{(i)}\}_{k=1}^{\infty}$ of mutually disjoint elements from $B(X_i)$ the relation

$$\nu_i(\bigcup_k Q_k^{(i)}) = \bigcup_k \nu_i(Q_k^{(i)})$$

holds. We call this the property of countable additivity of the function $\nu_i(\cdot)$;

2^0) the equality

$$\nu_i(X_i) = 1$$

takes place. We call this the property of normability.

Hence $\nu_i(Q^{(i)}) \leq 1 \ \forall Q^{(i)} \in B(X_i)$.

Denote the set of mixed strategies $\nu_i(\cdot)$ of the i-th player by $\{\nu_i\}$ $(i \in N)$.

Let $\delta(\cdot)$ be Dirac function. Then a measure of the form $\delta(x_i - x_i^*)(dx)$ is also a mixed strategy from the set $\{\nu_i\}$ $(i \in N)$. Note that the measure-products $\nu(dx) = \nu_1(dx_1) \cdot \ldots \cdot \nu_n(dx_n)$ determined in [13], [14] are the probability measures on the set X of situations (in pure strategies). Denote the set of probability measures $\nu(dx)$ by $\{\nu\}$. The measure $\nu(dx)$ is called a *situation in mixed strategies*.

Note that for constructing the measure-product $\nu(dx)$ we use the smallest σ-algebra $B(X)$ over $X_1 \times \ldots \times X_n = X$ such that $B(X)$ contains all Cartesian products $Q^{(1)} \times \ldots \times Q^{(n)}$, where $Q^{(i)} \in B(X_i)$ $(i \in N)$.

By [15], [16] the sets of all possible probability measures $\nu_i(dx_i)$ $(i \in N)$ and $\nu(dx)$ are weakly closed and weakly compact in itself sets. This means (for $\{\nu\}$) that from every infinite sequence $\{\nu^{(k)}\}$ $(k = 1, 2, \ldots)$ we can choose a subsequence $\{\nu^{(k_j)}\}$ $(j = 1, 2, \ldots)$ such that $\{\nu^{(k_j)}\}$ weakly converges to a measure $\nu^{(0)}(\cdot) \in \{\nu\}$. In other words, for any scalar function $\varphi(x)$ which is continuous on X, we have

$$\lim_{j\to\infty} \int\limits_X \varphi(x)\nu^{(k_j)}(dx) = \int\limits_X \varphi(x)\nu^0(dx)$$

and $\nu^{(0)}(\cdot) \in \{\nu\}$.

Since $\varphi(x)$ is continuous, the integrals (mathematical expectations) $\int\limits_X \varphi(x)\nu(dx)$ exist. By Fubini's theorem we have

$$\int\limits_X \varphi(x)\nu(dx) = \int\limits_{X_1} \ldots \int\limits_{X_n} \varphi(x)\nu_n(dx_n)\ldots\nu_1(dx_1),$$

where the order of integration can be changed.

3.3. **Mixed extension of the game (8).** We put into correspondence to the "game of guarantees"in pure strategies (8) its mixed extension

$$\langle N, \{\nu_i\}_{i\in N}, \{f_i[\nu] = \int\limits_X f_i[x]\nu(dx)\}_{i\in N}\rangle. \tag{12}$$

Here (as in (8)) N is a set of players' numbers, $\{\nu_i\}$ is a set of mixed strategies $\nu_i(\cdot)$ of the i-th player $(i \in N)$. In the game (12) each i-th player chooses his own strategy $\nu_i(\cdot) \in \{\nu_i\}$. As a result the situation $\nu(\cdot) \in \{\nu\}$ in mixed strategies is composed. Further we introduce the payoff function (mathematical expectation) $f_i[\nu] = \int\limits_X f_i[x]\nu(dx)$ of i-th player on the set $\{\nu\}$.

For the game (12) the situation in mixed strategies $\nu^{\mathrm{v}}(\cdot) \in \{\nu\}$ satisfies the *Berge equilibrium condition* if

$$\max_{\nu_{N\setminus\{i\}}(\cdot)\in\{\nu_{N\setminus\{i\}}\}} f_i[\nu\|\nu_i^{\mathrm{v}}] = f_i[\nu^{\mathrm{v}}] \quad (i \in N), \tag{13}$$

where

$$\nu_{N\setminus\{i\}}(dx_{N\setminus\{i\}} = \nu_1(dx_1)\ldots\nu_{i-1}(dx_{i-1})\nu_{i+1}(dx_{i+1})\ldots\nu_n(dx_n),$$

$$[\nu\|\nu_i^{\mathrm{v}}] = [\nu_1(dx_1)\ldots\nu_{i-1}(dx_{i-1})\nu_i^{\mathrm{v}}(dx_i)\nu_{i+1}(dx_{i+1})\ldots\nu_n(dx_n)],$$

$$\nu^{\mathrm{v}}(dx) = \nu_1^{\mathrm{v}}(dx_1)\ldots\nu_n^{\mathrm{v}}(dx_n).$$

For the game (12) the condition (13) determines the analogue of Berge equilibrium situation x^{v} satisfying (5). Denote the set of situations in mixed strategies $\nu^{\mathrm{v}}(\cdot) \in \{\nu\}$ satisfying (13) by $\{\nu^{\mathrm{v}}\}$.

3.4. **Properties of situations in mixed strategies satisfying the Berge equilibrium condition.**

3.4.1. *Weak compactness in itself of the set* $\{\nu^v\}$*.* We establish the weak compactness in itself of the subset $\{\nu^{\mathrm{v}}\} \subset \{\nu\}$.

Assume that $\varphi[x]$ is an arbitrary continuous on X scalar function. Suppose that the elements $\nu^{(k)}(\cdot)$ $(k = 1, 2, \ldots)$ of the infinite sequence $\{\nu^{(k)}(\cdot)\}_{k=1}^{\infty}$ belong to the set $\{\nu^{\mathrm{v}}\}$. Then, since $\{\nu^{\mathrm{v}}\} \subset \{\nu\}$, it follows that $\{\nu^{(k)}\}_{k=1}^{\infty} \subset \{\nu\}$. As $\{\nu\}$ is weakly compact in itself (see Subsection 3.2) there exists a subsequence $\{\nu^{(k_j)}(\cdot)\}_{j=1}^{\infty}$ and a measure $\nu^0(\cdot) \in \{\nu\}$ such that

$$\lim_{j\to\infty} \int\limits_X \varphi[x]\nu^{(k_j)}(dx) = \int\limits_X \varphi[x]\nu^0(dx).$$

Now we prove the validity of the inclusion $\nu^0(\cdot) \in \{\nu^{\mathrm{v}}\}$. Assume *the contrary.* Then for a rather large j there exists a number $i \in N$ and a situation $\bar{\nu}(\cdot) \in \{\nu\}$ such that $f_i[\bar{\nu}\|\nu_i^{k_j}] > f_i[\nu^{k_j}]$. This inequality contradicts the inclusion $\{\nu^{(k_j)}(\cdot)\}_{j=1}^{\infty} \subset \{\nu^{\mathrm{v}}\}$.

Thus for the game (12) we obtained weak compactness in itself of the set of situations $\{\nu^{\mathrm{v}}\}$.

Compactness of the set $f[\{\nu^{\mathrm{v}}\}] = \bigcup\limits_{\nu(\cdot)\in\{\nu^{\mathrm{v}}\}} f[\nu]$ (n-vector $f = (f_1, \ldots, f_n)$) in the criteria space $\mathbb{R}^n$ can be established in the same way.

3.4.2. *Auxiliary property 1.* Consider scalar functions (9) $\varphi_i(x, z) = f_i[x \parallel z_i] - f_i[z]$ and $\varphi(x, z) = \max\limits_{i\in N} \varphi_i(x, z)$. According to Subsection 2.3 it follows that if $f_i[x]$ $(i \in N)$ are continuous and the set $X \subset \mathbb{R}^l$ of situations x is compact, then $\varphi(x, z)$ is determined and continuous on $X \times X$.

We have $\varphi_i(x,z) \le \varphi(x,z) = \max_{i\in N} \varphi_i(x,z)$. Integrating the both parts of this inequality by an arbitrary measure-product $\mu(dx)\nu(dz)$, where $\mu(\cdot) \in \{\nu\}$ and $\nu(\cdot) \in \{\nu\}$, we get

$$\int\limits_{X\times X} \varphi_i(x,z)\mu(dx)\nu(dz) \le \int\limits_{X\times X} \max_{i\in N} \varphi_i(x,z)\mu(dx)\nu(dz)$$

for all $i \in N$. Therefore

$$\max_{i\in N} \int\limits_{X\times X} \varphi_i(x,z)\mu(dx)\nu(dz) \le \int\limits_{X\times X} \max_{i\in N} \varphi_i(x,z)\mu(dx)\nu(dz).$$

Taking into account the form of $\varphi_i(x,z)$ from (9) we have

$$\max_{i\in N} \int\limits_{X\times X} (f_i[x \parallel z_i] - f_i[z])\mu(dx)\nu(dz) \le \int\limits_{X\times X} \max_{i\in N} (f_i[x \parallel z_i] - f_i[z])\mu(dx)\nu(dz). \quad (14)$$

Remark. The inequality (14) is a generalization of the well-known property: maximum of sum do not exceed sum of maxima.

3.4.3. *Auxiliary property 2.* Now consider an auxiliary two-person zero-sum game

$$\Gamma_2 = \langle \{1,2\}, \{X_1 = X, X_2 = X\}, \varphi(x,z)\rangle.$$

In the game Γ_2 the set of strategies x of the first player is $X_1 = X$, the set of strategies z of the second player is $X_2 = X$. The payoff function $\varphi(x,z)$ of the first player is of the form (9). The aim of the first player is to choose his strategy $x \in X$ such that to get the largest possible value of the payoff function $\varphi(x,z)$. The aim of the second player is to choose a strategy $z \in X$ such that the function $\varphi(x,z)$ takes the least possible value.

The solution of the game Γ_2 is a saddle point $(x^0, z^0) \in X \times X$. It satisfies the relation

$$\varphi(x, z^{\text{v}}) \le \varphi(x^0, z^{\text{v}}) \le \varphi(x^0, z)$$

for $\forall x \in X$ and $\forall z \in X$.

Now assign a mixed extension

$$\widetilde{\Gamma}_2 = \langle \{1,2\}, \{\nu\}, \{\mu\}, \varphi(\nu,\mu)\rangle$$

for the game Γ_2.

Here $\{\nu\}$ is the set of mixed strategies $\nu(\cdot)$ of the first player,$\{\mu\} = \{\nu\}$ is the set of mixed strategies $\mu(\cdot)$ of the second player, the payoff function (mathematical expectation) of the first player is $\varphi(\nu,\mu) = \int\limits_{X\times X} \varphi_i(x,y)\mu(dx)\nu(dy)$.

The solution of the game $\widetilde{\Gamma}_2$ is a saddle point (ν^0, μ^{v}), where (ν^0, μ^{v}) is defined by the inequalities

$$\varphi(\nu, \mu^{\text{v}}) \le \varphi(\nu^0, \mu^{\text{v}}) \le \varphi(\nu^0, \mu) \quad (15)$$

for all $\nu(\cdot) \in \{\nu\}$ and $\mu(\cdot) \in \{\nu\}$. This solution is called a saddle point in mixed strategies for the game Γ_2.

Glicksberg proved in 1952 the theorem of existence of Nash equilibrium in mixed strategies for a non-cooperative game with $n \geq 2$ players [17]. For the special case of a non-cooperative game of $n \geq 2$ players, namely for a two-person zero-sum game $\widetilde{\Gamma}_2$, this theorem implies the following proposition.

Proposition. Suppose that the set $X \subset \mathbb{R}^l$ is compact and the payoff function of the first player $\varphi(x, z)$ is continuous on $X \times X$ in the game Γ_2. Then there exists a solution $(\nu^0, \mu^{\mathrm{v}})$ satisfying (15) in the game Γ_2, i.e. there exists a saddle point in mixed strategies for the game Γ_2.

Taking into account (9) we can present the inequalities (15) as

$$\int\limits_{X\times X} \max_{i\in N}(f_i[x \parallel z_i] - f_i[z])\mu(dx)\nu^{\mathrm{v}}(dz) \leq \int\limits_{X\times X} \max_{i\in N}(f_i[x \parallel z_i] - f_i[z])\mu^0(dx)\nu^{\mathrm{v}}(dz) \leq$$

$$\leq \int\limits_{X\times X} \max_{i\in N}(f_i[x \parallel z_i] - f_i[z])\mu^0(dx)\nu(dz) \tag{16}$$

for all $\mu(\cdot) \in \{\nu\}$ and $\nu(\cdot) \in \{\nu\}$. Setting $\nu(dz) = \mu^0(dx)$ the equality

$$\varphi(\mu^0, \nu) = \int\limits_{X\times X} \max_{i\in N}(f_i[x \parallel z_i] - f_i[z])\mu^0(dx)\nu(dz)$$

implies

$$\varphi(\mu^0, \nu) = 0.$$

Hence, taking into account (16) we obtain

$$\int\limits_{X\times X} \max_{i\in N}(f_i[x \parallel z_i] - f_i[z])\mu(dx)\nu^{\mathrm{v}}(dz) \leq 0.$$

By (14) we get

$$\max_{i\in N} \int\limits_{X\times X} (f_i[x \parallel z_i] - f_i[z])\mu(dx)\nu^{\mathrm{v}}(dz) \leq 0,$$

$$\max_{i\in N} \left[\int\limits_{X\times X} f_i[x \parallel z_i]\mu(dx)\nu^{\mathrm{v}}(dz) - \int\limits_{X\times X} f_i[z]\mu(dx)\nu^{\mathrm{v}}(dz) \right] \leq 0.$$

Then

$$\int\limits_{X\times X} f_i[x \parallel z_i]\mu(dx)\nu^{\mathrm{v}}(dz) \leq \int\limits_{X\times X} f_i[z]\mu(dx)\nu^{\mathrm{v}}(dz) \;\; \forall \mu(\cdot) \in \{\nu\}.$$

Taking into account normalization of probability measure $\mu(\cdot)$ (see Subsection 3.2) we have $\int\limits_X \mu(dx) = 1$. Then the previous inequality implies

$$\int\limits_{X\times X} f_i[x \parallel z_i]\mu(dx)\nu^{\mathrm{v}}(dz) \leq \int\limits_{X\times X} f_i[z]\mu(dx)\nu^{\mathrm{v}}(dz) \;\; \forall i \in N.$$

Using notations from (12), taking into account $f_i[\mu \parallel \nu_i] = \int\limits_{X\times X} f_i[x \parallel z_i]\mu(dx)\nu^{\mathrm{v}}(dz)$, we get

$$f_i[\mu \parallel \nu_i^{\mathrm{y}}] \leq f_i[\nu^{\mathrm{v}}] \quad (i \in N),$$

i.e. condition (13) holds.

Thus, if the sets $X_i \subset \mathbb{R}^{l_i}$ $(i \in N)$ are compact and the payoff functions $f_i[x]$ of every i-th player are continuous on X in the game (8), then there exists a situation in mixed strategies $\nu^{\mathrm{v}}(\cdot) \in \{\nu\}$ satisfying Berge equilibrium condition (5) in the game (8).

4. Existence

4.1. **The notion of strong guaranteed Berge equilibrium in mixed strategies.** In this section we present the main result of present paper. We establish existence of a strong guaranteed Berge equilibrium in mixed strategies for the game

$$\Gamma_N = \langle N, \{X_i\}_{i\in N}, Y^X, \{f_i(x,y)\}_{i\in N}\rangle.$$

For this we assign a quasimixed extension

$$\widetilde{\Gamma}_N = \langle N, \{\nu_i\}_{i\in N}, Y^X, \{f_i(\nu,y)\}_{i\in N}\rangle$$

to the game Γ_N.

Recall that in Γ_N the sets $X_i \subset \mathbb{R}^{l_i}$ $(i \in N)$ are compact ones. $N = \{1,\dots,n\}$ is the set of players' numbers in the game $\widetilde{\Gamma}_N$ (as in Γ_N). In $\widetilde{\Gamma}_N$ every i-th player $(i \in N)$ can use both pure strategies $x_i \in X_i \subset \mathbb{R}^{l_i}$ $(i \in N)$ (as in Γ_N) and mixed strategies (probability measures) $\nu(\cdot)$ determined on the Borel σ-algebra $B(X_i)$ over the compact set $X_i \subset \mathbb{R}^{l_i}$ (see subsection 3.2). Y^X is the set of uncertainties $y(x) : X \to Y$, $Y \subset \mathbb{R}^m$. The payoff function of the i-th player is of the form

$$f_i(\nu,y) = \int\limits_X f_i(x,y)\nu(dx). \tag{17}$$

Similarly to Subsection 2.1, we introduce the notion of a strong guaranteed equilibrium $(\bar{\nu}^V, \bar{f}^s) \in \{\nu\} \times \mathbb{R}^n$ $(f = (f_1,\dots,f_n))$ in mixed strategies for the game Γ_N. For this we use three stages:

Stage 1. Taking into account the relation

$$\min_{y\in Y} f_i(x,y) = f_i(x, y^{(i)}(x)) = f_i[x] \ \ \forall x \in X \ \ (i \in N)$$

we construct n vector-functions $y^{(i)}(x) \in Y^X$.

Stage 2. For the non-cooperative "game of guarantees"of n persons

$$\langle N, \{\nu_i\}_{i\in N}, \{f_i[\nu]\}_{i\in N}\rangle, \tag{18}$$

where the payoff function of the i-th player is defined by the equality $f_i[\nu] = \int\limits_X f_i[x]\nu(dx)$ $(i \in N)$, we find a set $\{\nu^{\text{v}}\} \subset \{\nu\}$ of situations in mixed strategies $\nu^{\text{v}}(\cdot)$ such that

$$\max_{\nu_{N\setminus\{i\}}(\cdot)\in\{\nu_{N\setminus\{i\}}\}} f_i[\nu\|\nu_i^{\text{v}}] = f_i[\nu^{\text{v}}] \ \ (i \in N). \tag{19}$$

Here the measure-product $\nu_{N\setminus\{i\}}(dx_{N\setminus\{i\}}) = \nu_1(dx_1)\dots\nu_{i-1}(dx_{i-1})\nu_{i+1}(dx_{i+1})\dots\nu_n(dx_n)$, i.e. $\nu^{\text{v}}(\cdot)$ satisfies the Berge equilibrium condition in the game (18).

Stage 3. For the n-criteria problem

$$\langle\{\nu^{\text{v}}\}, \{f_i[\nu]\}_{i\in N}\rangle$$

construct a Slater maximal solution $\bar{\nu}^{\text{v}}(\cdot) \in \{\nu\}$ such that for any $\nu^{\text{v}}(\cdot) \in \{\nu^{\text{v}}\}$ the system of strict inequalities

$$f_i[\nu] > f_i[\bar{\nu}^{\text{v}}] = \bar{f}_i^s \quad (i \in N)$$

is inconsistent.

Then the couple $(\bar{\nu}^{\text{v}}, \bar{f}^s = (\bar{f}_1^s, \dots, \bar{f}_n^s))$ is called a *Berge strong guaranteed equilibrium in mixed strategies for the game* Γ_N. The situation in mixed strategies $\bar{\nu}^{\text{v}}(\cdot)$ is called a *guaranteeing situation*; n-vector $\bar{f}^s$ is called a *vector guarantee.*

4.2. **Proof of existence.** Assume that the following conditions hold for the game Γ_N:

(1^0) the sets $X_i \subset \mathbb{R}^{l_i}$ $(i \in N)$ and $Y \subset \mathbb{R}^m$ are compact and Y is convex as well;

(2^0) the payoff functions $f_i(x, y)$ $(i \in N)$ are continuous on $X \times Y$ $(X = \prod\limits_{i\in N} X_i)$;

(3^0) for any $x \in X$ the payoff functions $f_i(x, y)$ $(i \in N)$ are strictly convex in $y \in Y$, i.e. for all $\lambda = \mathit{const} \in (0, 1)$ and $y^{(j)} \in Y$ $(j = 1, 2)$ the following strict inequalities hold

$$f_i(x, \lambda_i y^{(1)} + (1 - \lambda_i)y^{(2)}) < \lambda_i f_i(x, y^{(1)}) + (1 - \lambda_i)f_i(x, y^{(2)}) \quad (i \in N).$$

We prove that fulfillment of conditions $1^0 - 3^0$ provides existence of strong guaranteed equilibrium of Berge in mixed strategies in the game Γ_N. In other words, we are going to prove that conditions $1^0 - 3^0$ yield existence of a couple $(\bar{\nu}(\cdot), \bar{f}^s)$ satisfying the requirements of Stages 1-3 from Subsection 4.1.

Stage 1. By Subsection 1.1 fulfillment of conditions $1^0 - 3^0$ yields existence of a continuous m-vector function

$$y^{(i)}(x) : \min_{y\in Y} f_i(x, y) = f_i(x, y^i(x)) = f_i[x] \ \ \forall x \in X \ \ (i \in N). \tag{20}$$

Note that the functions $f_i[x] = f_i(x, y^{(i)}(x))$ $(i \in N)$ are continuous on X (as a superposition of continuous functions $f_i(x, y)$ and $y^{(i)}(x)$). By (20) for every $x \in X$ we have

$$f_i[x] \le f_i(x, y) \ \ \forall y \in Y (i \in N). \tag{21}$$

Integrating the both parts of the inequality (21) by an arbitrary measure $\nu(\cdot) \in \{\nu\}$ we get

$$f_i[\nu] = \int\limits_X f_i[x]\nu(dx) \le \int\limits_X f_i(x,y)\nu(dx) = f_i(\nu, y) \ \ \forall y \in Y \ \ (i \in N). \qquad (22)$$

Due to the inequality (21) we can assign the "game of guarantees"

$$\Gamma_g = \langle N, \{X_i\}_{i\in N}, \{f_i[x]\}_{i\in N}\rangle$$

to the game Γ_N.

Since for all $y \in Y$ we have

$$f_i[x] \le f_i(x,y) \ \ (i \in N),$$

then the vector guarantee $f[x] = (f_1[x], \dots, f_n[x])$ corresponds to each situation $x \in X$ in Γ_g. In the same way, due to the inequality (22) in the "game of guarantees"in mixed strategies

$$\widetilde{\Gamma}_g = \langle N, \{\nu_i\}_{i\in N}, \{f_i[\nu] = \int\limits_X f_i[x]\nu(dx)\}_{i\in N}\rangle$$

the vector guarantee $f[\nu] = (f_1[\nu], \dots, f_n[\nu])$ corresponds to each situation in mixed strategies $\nu(\cdot) \in \{\nu\}$.

Since inequalities (21) and (22) hold for all $i \in N$ then it follows that the guarantees $f[x]$ and $f[\nu]$ are the "smallest". This is a main reason to use the term "strong guaranteed".

Stage 2. Since the sets X_i $(i \in N)$ are compact and the function $f_i[x]$ is continuous on $X = \prod\limits_{i\in N} X_i$ (see conditions $1^0 - 2^0$ and Stage 1), taking into account Subsection 3.4.3, there exists a situation in mixed strategies $\nu^{\mathrm{v}}(\cdot)$ satisfying the requirement of Berge equilibrium (19) in the game $\widetilde{\Gamma}_g$. Therefore the set $\{\nu^{\mathrm{v}}\} \ne \emptyset$. By Subsection 3.4.1 the set $\{\nu^{\mathrm{v}}\}$ is weakly compact in itself, then the set of values of payoff functions

$$\Phi = f[\{\nu^{\mathrm{v}}\}] = \bigcup_{\nu(\cdot)\in\{\nu^{\mathrm{v}}\}} f[\nu^{\mathrm{v}}] \ \ (here f[\nu] = (f_1[\nu], \dots, f_n[\nu])) \qquad (23)$$

is a compact set in $\mathbb{R}^n$ (see the proof in Subsection 3.4.1).

Stage 3. Let $\alpha_i = const \ge 0$ and $\sum\limits_{i\in N} \alpha_i > 0$. Consider a linear convolution $\sum\limits_{i\in N} \alpha_i f_i$ determined on Φ (see (23)). Since $\sum\limits_{i\in N} \alpha_i f_i$ is continuous on Φ and taking into account Weierstrass theorem we get that there exists a constant n-vector $\bar{f}^s = (\bar{f}_1^s, \dots, \bar{f}_n^s)) \in \Phi$ such that $\max\limits_{f\in\Phi} \sum\limits_{i\in N} \alpha_i f_i = \sum\limits_{i\in N} \alpha_i \bar{f}_i^s$. Due to Karlin's Lemma [19] the alternative $\bar{f}^s$ is maximal by Slater in the n-criteria problem

$$\langle \Phi, \{f_i\}_{i\in N}\rangle,$$

i.e. for any $f \in \Phi$ the system of inequalities

$$f_i > \bar{f}_i^s \quad (i \in N)$$

is inconsistent. Taking into account the construction way of the set Φ (see (23)) one can state that there exists a situation $\bar{\nu}^s(\cdot) \in \{\nu\}$ such that $\bar{f}^s = f[\bar{\nu}^s]$. This situation in mixed strategies $\bar{\nu}^s(\cdot)$ is Slater maximal in n-criteria problem $\langle\{\nu^{\mathrm{v}}\}, \{f_i[\nu]\}_{i\in N}\rangle$. According to the definition from Subsection 4.1 the couple $(\bar{\nu}^{\mathrm{v}}, \bar{f}^s)$ is a Berge strong guaranteed equilibrium in mixed strategies for the game Γ_N.

Thus, if for the game Γ_N the following conditions

- the sets $X_i \subset \mathbb{R}^{l_i}$ $(i \in N)$ and $Y \subset \mathbb{R}^m$ are compact;
- the set Y is convex;
- the scalar payoff functions $f_i(x, y)$ $(i \in N)$ are continuous on $X \times Y$;
- for any $x \in X$ the payoff functions $f_i(x, y)$ $(i \in N)$ are strictly convex in $y \in Y$

hold, then there exists a Berge strong guaranteed solution (BVSGS) in mixed strategies in the game Γ_N.

Remark. The "game sense"of the notion of BVSGS is as follows. Every situation has a corresponding vector guarantee. Among these guarantees we have to select the ones which correspond to the Berge equilibrium situations. Then from such guarantees a Slater maximal guarantee (with respect to the selected guarantees) has to be chosen. The couple (the equilibrium situation and the corresponding vector guarantee) is offered as a "good"solution (BVSGS) for the game Γ_N. In fact, whatever uncertainty is realized in the game Γ_N, the players (using the situation from BVSGS) provide themselves "the largest"guaranteed payoffs. For each player this guaranteed payoff coincides with the corresponding component of the vector guarantee.

5. Conclusions

In this paper two new basic results of game theory have been established. These results concern Berge equilibrium (see the review [18]).

First, for the non-cooperation game

$$\Gamma_g = \langle N, \{X_i\}_{i\in N}, \{f_i[x]\}_{i\in N}\rangle,$$

where $N = \{1, \ldots, n\}$ is the set of players' numbers, the set of strategies x_i of the i-th player is $X_i \subset \mathbb{R}^{l_i}$ $(i \in N)$, the situations $x = (x_1, \ldots, x_n) \in X = \prod_{i\in N} X_i \subset \mathbb{R}^l$, the i-th player payoff function $f_i[x]$ is determined on X, the following proposition have been obtained.

Proposition 1. If X_i are compact and $f_i[x]$ are continuous on X_i $(i \in N)$ then in the game Γ_g there exist the situations in mixed strategies $\bar{\nu}^{\mathrm{v}}(\cdot) \in \{\nu\}$ satisfying the Berge equilibrium condition (19) such that $\bar{\nu}^{\mathrm{v}}(\cdot)$ is Slater maximal with respect to all other situations satisfying the Berge equilibrium condition.

Second, we considered a non-cooperation game of n persons under uncertainty

$$\Gamma_N = \langle N, \{X_i\}_{i\in N}, Y^X, \{f_i(x,y)\}_{i\in N}\rangle,$$

where N, x_i, X_i, x, X are the same as in Γ_g, Y^X is the set of uncertainties $y(x) : X \to$ $\to Y \subset \mathbb{R}^m$, on the set $X \times Y$ the payoff function $f_i(x,y)$ of any i-th player is determined.

In subsection 4.1 for the game Γ_N we have introduced the notion of Berge strong guaranteed equilibrium in mixed strategies.

Proposition 2. Let in the game Γ_N the following conditions

– the sets $X_i \subset \mathbb{R}^{l_i}$ $(i \in N)$ and $Y \subset \mathbb{R}^m$ are compact;

– the set Y is convex;

– the scalar payoff functions $f_i(x,y)$ $(i \in N)$ are continuous on $X \times Y$;

– for any $x \in X$ the payoff functions $f_i(x,y)$ $(i \in N)$ are strictly convex in $y \in Y$

hold, then there exists a Berge strong guaranteed solution in mixed strategies in the game Γ_N.

References

[1] Berge C. *Théorie Générale des Jeux a n Personnes Games* // Paris: Gauthier-Villars, 1957.

[2] Shubik M. *Review: The general theory of n-person games by Claude Berge* // Econometrica 29(4) (1961), p. 821.

[3] Zhukovskiy V.I. and Chikrii A.A. *Linear-Quadratic Differential Games* // Naukova Dumka, Kiev, 1994 (in Russian).

[4] Zhukovskiy V.I., Salukvadze M.E. and Vaisman K.S. *The Berge equilibrium* // Preprint, Institute of Control Systems, Tbilisi, 1994.

[5] Vaisman K.S. *Berge Equilibrium* // PhD Thesis, Saint-Petersburg State University, 1995 (in Russian).

[6] Vaisman K.S. *Berge equilibrium in one differential game, in: Complex Dynamical Systems*// Pskov Pedagogical Institute, Pskov, Russia, 1994, pp. 58-63 (in Russian).

[7] Vaisman K.S. *The Berge equilibrium for linear-quadratic differential game* // Multiple criteria problems under uncertainty: Abstracts of the 2-d International Workshop, Orekhovo-Zuevo, Russia, 1994, pp. 27-28.

[8] Vorob'ev N.N. *Game Theory for Economists-Cyberneticists* // Nauka, Moscow, 1985 (in Russian).

[9] Zhukovskiy V.I., Zhyteneva J.N. *Existence of Berge-Vaisman equilibrium* // International Scientific Journal "Spectral and Evolution Problems"18 (2008), 61-64 (in Russian).

[10] Morozov V.V. *Foundations of Game Theory* // Faculty of Computational Mathematics and Cybernetics, Lomonosov Moscow State University, Moscow, 2002 (in Russian).

[11] Morozov V.V., Sukharev A.G. and Fiodorov V.V. *Operations Research: Problems and Exercises* // Vys'shaya shkola, Moscow, 1986 (in Russian).

[12] Germayer Y.B. *Introduction to Operations Research* // Nauka, Moscow, 1971 (in Russian).

[13] Krasovskiy N.N. and Subbotin A.I. *Game-theoretical Control Problems.* Springer-Verlag, Berlin, 1988.

[14] Loeve M. *Probability Theory* // Princeton, D. Van Nostrand Company Inc., London, New York, Toronto, 1955.

[15] Dunford N. and Schwartz T. *Linear Operators* // Part 1, Interscience Publishers, New York and London, 1958.

[16] Lusternic L.A. and Sobolev V.I. *Elements of Functional Analysis* // Nauka, Moscow, 1969 (in Russian).

[17] Glicksberg I.I. *A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points* // Proc. Amer. Math. Soc., 3(1) (1952), 170-174.

[18] Colman A.M., Körner T.W., Musy O. and Tazdaït T. *Mutual support in games: Some properties of Berge equilibria* // Journal of Mathematical Psycology 55(2) (2011), 1-10.

[19] Karlin S. *Mathematical Methods and Theory in Games, Programming and Economics* // Pergamon Press, London-Paris, 1959.

[20] Zhukovskiy V.I. *Introduction to Differential Games under Uncertainty: Berge-Vaisman Equilibrium* // PRASAND, Moscow 2010 (in Russian).

Существование равновесия по Бержу в смешанных стратегиях

Для бескоалиционной игры n лиц при неопределенности формализуется понятие гарантированного решения, основанного на подходящей модификации максимина и равновесия по Бержу. Получены условия существования гарантированного решения в классе смешанных стратегий (вероятностных мер).

Ключевые слова: вероятностные меры, смешанная стратегия, слабая компактность, гарантия, равновесие по Бержу-Вайсману, равновесие по Нэшу, максимин.