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SMALL MOTIONS AND EIGENOSCILLATIONS OF A SYSTEM „FLUID – GAS" IN A BOUNDED REGION[1]

1. Introduction.

1.1. **To the history of the problem.** The problem on small motions of an ideal fluid in a partially filled vessel was a subject of numerous investigations at the second half of the 20$^{\text{th}}$ century. We mention here only monographs [1] – [4] corresponding to the case when a fluid is heavy and monographs [5] – [7] for the so-called capillary fluid, i.e., a fluid that moves under action not only gravity but surface tension on a free surface (zero – gravity conditions).

For capillary fluid static problems were studied in the first parts of monographs [5] – [7]. Small motions and eigen oscillations were considered in the second parts of [5] – [7] and in monographs [8] – [10]. Here authors used methods of functional analysis, the theory of differential equations in Hilbert space, spectral theory of operators and operator functions.

In the paper, we study a new class of problems where immovable container not partially filled by an incompressible fluid or the system of incompressible ones but the case when the first fluid is incompressible ideal and the second one is a barotropic gas. The first papers on this topics are published in works [11] – [16] and [17] – [20].

This paper is written on the base of Chapter 1 of phD – thesis [21] where a heavy ideal fluid and a gas was considered. Here we use the operator approach which is discribed in detail in [8] – [9] for the case of one ideal incompressible capillary fluid or for the case of a system of such fluids.

1.2. **Main results of the paper.** In Section 2, we formulate the statement of the problem on small motions and eigenoscillations of a system „ideal incompressible capillary fluid – gas". We consider preliminary an equilibrium state of the system and discribe the main parameters of the problem, in particular, parameters connected with surface tension and barotropic gas.

[1]The first author of the paper is gratefull to prof. V.A. Solonnikov for invitation to Ferrara University (Italy) at June, 2004, for joint collaborations.

After that we formulate the statement of the initial boundary value problem on small motions of a system „fluid – gas" (see (8) – (15)). We derive the law of full energy balance for classic solution of this problem (see (20)). The next step is connected with using method of orthogonal projecting of vector equations of the problem on subspaces of the spaces $\vec{L}_2(\Omega_1)$ and $\vec{L}_2(\Omega_2)$ for vector functions described displacement fields of a fluid and a gas. It gives us some trivial relations (see (31), (32), (41)) and nontrivial equations (see (33), (40) and boundary conditions) in subspaces of the spaces $\vec{L}_2(\Omega_1)$ and $\vec{L}_2(\Omega_2)$. This approach allows us to reformulate the initial boundary value problem (8) – (15) in a new form (see (55) – (61)) for finding of two scalar functions: displacement potentials Φ_1 and Φ_2 for a fluid and a gas.

We formulate also the problem on eigenoscillations of the system, i.e., on finding solutions of homogeneous problem depending in time t according to the law $\exp(\mathrm{i}\omega t)$ where ω is a frequency of oscillations. Then spectral problem (63) – (68) arises with spectral parameter $\lambda = \omega^2$. The statement of this problem contains the potential energy operator B_σ (see (51) and Lemma 1), and we suppose that investigated system is statically stable in linear approximation, i.e., the operator B_σ is positive definite (see (69)).

In Section 3, we investigate the problem on eigenoscillations on the base of auxiliary boundary value problems and corresponding Hilbert spaces and its equipments. We introduce the operators of these problems (Subsection 3.1) and transit to matrix operator equation (or the system of two operator equations) in orthogonal sum of Hilbert spaces (see (101), (102) and (107) – (109)). We study properties of entries of these operator matrices and on this base we prove the theorem on the structure of the spectrum and properties of eigenfunctions (Theorem 2).

In Section 4, we consider variation principles for eigenvalues (Theorems 128 – 130) and show that the variation principle in the form (165) is the most convenient in applications when we use Ritz method for calculations of eigenvalues.

In Section 5, we investigate the orthogonal basis properties of eigenfunctions and prove that these functions form an orthogonal basis in some Hilbert space (Theorems 6, 7). On the base of variation principles we consider also some limit cases (Subsection 5.3) connected with transit to one incompressible fluid (without of a gas), to the case, when a gas transforms to an ideal incompressible fluid, or to the case, when only one barotropic gas fills all the region. At last, we consider briefly the problem on surface and acoustic waves arising in our system „fluid – gas" (Subsection 5.4).

Section 6 is devoted to investigation on the problem of existence of strong (according to variable t) solutions to the initial boundary value problems in a vector and in a scalar forms (see (8) – (15) and (55) – (61)). We prove that our problem is reduced to investigation of Cauchy problem for some hyperbolic equation in Hilbert space. As a result, we prove the theorem on strong solvability of the initial boundary value problem for operator equation in orthogonal sum of Hilbert spaces (see (208) – (211), Theorem

9), for scalar problem (55) – (61) (Theorem 10) and for initial vector boundary value problem (8) – (15) (Theorem 11). At last, for strong and generalized solutions to these problem we prove the law of full energy balance (Theorem 12).

Further, using basis properties of eigenfunctions, we represent a strong (and formal) solution to problem (55) – (61) by Fourier series on eigenfunctions of the spectral problem (63) – (68) (Subsection 6.4).

If condition of the static stability in linear approximation is not fulfilled and instead of property (69) the operator B_σ of potential energy is only bounded from below with lower bound negative then considered system „fluid – gas" is unstabled. In Subsection 6.5 we prove (Theorems 13, 14) that in the case our system is dynamical unstable.

At last, in Subsection 6.6 we briefly consider a problem on small motions and eigenoscillations of a system „fluid – gas" for the case when surface tension do not taken into account, i.e., for a heavy fluid. This problem is considered more explicitly in work [21].

2. The statement of the problem.

In this section, the mathematical statement of the initial boundary value problem on small motions and eigenoscillations of a hydrosystem „fluid – gas" is formulated. We write down the equations, boundary value and initial conditions. The transition from vector problem to scalar one is realized. The corresponding spectral problem is also formulated.

2.1. **Equations of the initial boundary value problem.**

Consider a hydrodynamical system consisting of two nonmixing ideal fluids. The first of them is incompressible and the second one is compressible that is a gas. We suppose that fluids fulfill an arbitrary region $\Omega \in \mathbb{R}^3$ and we will take into account gravitation forces with acceleration $\vec{g}$ and surface tension. At equilibrium state a lower fluid is incompressible and has a constant density $\rho_1 > 0$ and upper compressible fluid (gas) has a density $\rho_2 < \rho_1$. The lower fluid occupides a region $\Omega_1 \subset \Omega$ bounded by a part S_1 of the rigid wall $S := \partial\Omega$ and by the surface Γ which is an equilibrium one dividing a fluid and a gas. Respectively, a gas occupies a region $\Omega_2 = \Omega \backslash \Omega_1$ bounded by the surface Γ and by a part $S_2 = S \backslash S_1$ of the rigid wall S.

We introduce the cartesian coordinate system $Ox_1x_2x_3$ by such a way that $\vec{g} = -g\vec{e_3}$, where $\vec{e_i}$ is an ort of the axis Ox_i, $i = 1, 2, 3$.

At the equilibrium state pressures in a fluid and in a gas are changed along the vertical axis Ox_3 and have the form

$$P_{i,0}(x) = P_{i,0}(x_3) = -\rho_i g x_3 + c_i, \quad \text{in } \Omega_i, \quad i = 1, 2, \tag{1}$$

where c_i are constants. At the equilibrium surface Γ the Laplace condition for the jump of pressures must be fulfilled:

$$P_{1,0} - P_{2,0} = -\sigma(k_1 + k_2) \quad \text{on } \Gamma. \tag{2}$$

Here $\sigma > 0$ is a coefficient of surface tension on the boundary „fluid – gas", k_1 and k_2 are the main curvatures of Γ. On the contour $\partial\Gamma$ the condition of Dupre – Yung must be valid:

$$\sigma\cos\delta = \sigma_1 - \sigma_0, \tag{3}$$

where δ is a wetting angle, $0 < \delta < \pi$, $\sigma_1 > 0$ is a corresponding coefficient on the boundary „fluid – rigid wall" and $\sigma_0 > 0$ is a corresponding one on the boundary „gas – rigid wall".

We suppose that the volume V of the fluid is given, that is

$$\int\limits_{\Omega_1} d\Omega = V, \tag{4}$$

and then conditions (1) – (4) allows us to find an equilibrium surface Γ and regions Ω_1 and Ω_2 (see, for instance, the monographs [5] – [7]).

Suppose that this static problem is solved and consider small motions of the hydrosystem near the equilibrium state. We introduce unknown functions $\vec{w_i}(t,x)$, $i = 1,2$, $x \in \Omega_i$, which are displacements fields in a fluid and a gas, and dynamic pressures $p_i(t,x)$ which are differences between full pressures $P_i(t,x)$ and static ones $P_{i,0}(x_3)$.

Let $\widetilde{\rho_2}(t,x)$ be a density of a moving gas. Then $\widetilde{\rho}_2 = \rho_2 + \eta(t,x)$, where $\eta(t,x)$ is a new anknown function. For barotropic gas we have (see, for instance, [22], pp. 299-300)

$$p_2 = \left(\frac{dP_2}{d\widetilde{\rho_2}}\right)_{\widetilde{\rho}_2 = \rho_2} \cdot \eta =: c^2\eta, \tag{5}$$

where c^2 is a squared sound velocity. Therefore from the continuouty equation (with velocity field $\vec{u_2} = \partial\vec{w_2}/\partial t$),

$$\frac{\partial\widetilde{\rho_2}}{\partial t} + \mathrm{div}\left(\widetilde{\rho_2}\frac{\partial\vec{w_2}}{\partial t}\right) = 0,$$

one can find after linearization the relation

$$\frac{\partial}{\partial t}\left(p_2 + c^2\rho_2 \mathrm{div}\vec{w_2}\right) = 0. \tag{6}$$

For $\vec{w_2}(t,x) \equiv \vec{0}$ we must have $p_2(t,x) \equiv 0$, and then from (6) we receive

$$p_2 + c^2\rho_2 \mathrm{div}\vec{w_2} = 0 \quad \text{in } \Omega_2. \tag{7}$$

(If $c^2 \longrightarrow \infty$ then it follows from (7) that $\mathrm{div}\,\vec{w_2} = 0$, that is the second fluid becomes incompressible.)

Let us write down equations, boundary value and initial conditions of the problem on small motions of a hydrosystem „ideal fluid – gas". With account of (7) we have

$$\rho_1\frac{\partial^2\vec{w_1}}{\partial t^2} + \nabla p_1 = \rho_1\vec{f}, \quad \mathrm{div}\,\vec{w_1} = 0 \quad (\text{in } \Omega_1), \tag{8}$$

$$\rho_2\frac{\partial^2\vec{w_2}}{\partial t^2} + \nabla p_2 = \rho_2\vec{f}, \quad p_2 + c^2\rho_2\,\mathrm{div}\,\vec{w_2} = 0 \quad (\text{in } \Omega_2), \tag{9}$$

$$\vec{w_1} \cdot \vec{n} = 0 \quad (\text{on } \ S_1), \quad \vec{w_2} \cdot \vec{n} = 0 \quad (\text{on } \ S_2), \tag{10}$$

$$\vec{w_1} \cdot \vec{n} = \vec{w_2} \cdot \vec{n} =: \zeta \quad (\text{on } \ \Gamma), \quad \int\limits_{\Gamma} \zeta d\Gamma = 0, \tag{11}$$

$$p_1 - p_2 = \mathcal{L}_\sigma \zeta := a_\sigma \zeta - \sigma \Delta_\Gamma \zeta \quad (\text{on } \ \Gamma), \tag{12}$$

$$a_\sigma = a_\sigma(x) := (\rho_1 - \rho_2) g \cos(\vec{n}, \vec{e_3}) - \sigma(k_1^2 + k_2^2), \quad x \in \Gamma, \tag{13}$$

$$\frac{\partial \zeta}{\partial e} + \chi \zeta = 0 \quad (\text{on } \ \partial\Gamma), \quad \chi := \frac{k_\Gamma - k_S \cos\delta}{\sin \delta}, \tag{14}$$

$$\vec{w_i}(0, x) = \vec{w_i}^0(x), \quad \frac{\partial \vec{w_i}}{\partial t}(0, x) = \vec{w_i}^1(x), \quad i = 1, 2. \tag{15}$$

Here the first equations in (8) and (9) are the linearized Euler equations for displacements fields $\vec{w_i}$ and dynamic pressures p_i; $\vec{f} = \vec{f}(t, x)$ is a known function of an additional external small field of mass forces: $\vec{F} = \vec{g} + \vec{f}$; $\vec{n}$ is an external unique normal to Ω_1; $\zeta = \zeta(t, x) \quad (x \in \Gamma)$ is a displacement (along the normal $\vec{n}$) of a moving surface $\Gamma = \Gamma(t)$ in process of oscillations; Δ_Γ is a Laplace – Beltrami operator, acting on Γ; a_σ is a known function that is defined by the equilibrium state; $\vec{e}$ is a unique normal vector to $\partial\Gamma$ in the plane tangential to Γ on $\partial\Gamma$; k_Γ and k_S are the curvatures of Γ and S in a cross section of Γ and S by the plane that is perpendicular to $\partial\Gamma$. (One can see the derivation of conditions (12) – (14) in [9], pp. 201 – 203.) The second condition in (8) is a condition of incompressibility for the displacement field $\vec{w_1}$, the second condition in (9) is a condition of compressibility for barotropic gas (see (7)). Conditions (10) are so-called nonleaking conditions on the rigid wall S. The first condition (11) is a kinematic condition on Γ, and the second one in (11) is a condition of volume conservation of the fluid. Condition (12) is a linearized condition for pressure jump on moving surface $\Gamma(t)$; the corresponding nonlinear condition has the same form as (2). Condition (14) is a corollary of the fact, that wetting angle δ, $0 < \delta < \pi$, does not changed in process of oscillations (see [9], p. 201 – 203).

Thus, the problem on small motions of a hydrosystem „fluid – gas" consist of finding displacements fields $\vec{w_i}(t, x)$ and pressures fields $p_i(t, x)$ from equations, boundary value and initial conditions (8) – (15).

2.2. **The law of full energy balance.** We will derive, on the base of equations, boundary value and initial conditions of problem (8) – (15), the law of full energy balance for investigated hydrodynamical system. This system is conservative, then, if additional external forces are absent ($\vec{f}(t, x) \equiv \vec{0}$), it will be the law of full energy conservation.

Suppose that problem (8) – (15) has a classical solution, that is, all unknown functions and its derivatives that are located in equations, boundary value and initial conditions, are continuous functions.

From the first equation (8) and with account of the second one and the first condition (10) we have

$$\rho_1 \int\limits_{\Omega_1} \frac{\partial^2 \vec{w}_1}{\partial t^2} \cdot \frac{\partial \vec{w}_1}{\partial t}\, d\Omega_1 = \frac{d}{dt}\left(\frac{1}{2}\rho_1 \int\limits_{\Omega_1} \left|\frac{\partial \vec{w}_1}{\partial t}\right|^2 d\Omega_1\right) = -\int\limits_{\Omega_1} \nabla p_1 \cdot \frac{\partial \vec{w}_1}{\partial t}\, d\Omega_1 +$$

$$+\rho_1 \int\limits_{\Omega_1} \vec{f} \cdot \frac{\partial \vec{w}_1}{\partial t}\, d\Omega_1 = -\int\limits_{\Omega_1} \mathrm{div}\left(p_1 \frac{\partial \vec{w}_1}{\partial t}\right) d\Omega_1 + \rho_1 \int\limits_{\Omega_1} \vec{f} \cdot \frac{\partial \vec{w}_1}{\partial t}\, d\Omega_1 =$$

$$= -\int\limits_{\partial\Omega_1} p_1 \frac{\partial \vec{w}_1}{\partial t} \cdot \vec{n}\, dS + \rho_1 \int\limits_{\Omega_1} \vec{f} \cdot \frac{\partial \vec{w}_1}{\partial t}\, d\Omega_1 = -\int\limits_{\Gamma} p_1 \frac{\partial \vec{w}_1}{\partial t} \cdot \vec{n}\, d\Gamma + \rho_1 \int\limits_{\Omega_1} \vec{f} \cdot \frac{\partial \vec{w}_1}{\partial t}\, d\Omega_1.$$

From the first equation (9) with account of the second one and the second condition (10) we derive analogously

$$\rho_2 \int\limits_{\Omega_2} \frac{\partial^2 \vec{w}_2}{\partial t^2} \cdot \frac{\partial \vec{w}_2}{\partial t}\, d\Omega_2 = \frac{d}{dt}\left(\frac{1}{2}\rho_2 \int\limits_{\Omega_2} \left|\frac{\partial \vec{w}_2}{\partial t}\right|^2 d\Omega_2\right) = -\int\limits_{\Omega_2} \nabla p_2 \cdot \frac{\partial \vec{w}_2}{\partial t}\, d\Omega_2 +$$

$$+\rho_2 \int\limits_{\Omega_2} \vec{f} \cdot \frac{\partial \vec{w}_2}{\partial t}\, d\Omega_2 = -\int\limits_{\Omega_2} \mathrm{div}\left(p_2 \frac{\partial \vec{w}_2}{\partial t}\right) d\Omega_2 - \frac{1}{\rho_2 c^2} \int\limits_{\Omega_2} p_2 \cdot \frac{\partial p_2}{\partial t}\, d\Omega_2 +$$

$$+\rho_2 \int\limits_{\Omega_2} \vec{f} \cdot \frac{\partial \vec{w}_2}{\partial t}\, d\Omega_2 = \int\limits_{\Gamma} p_2 \frac{\partial \vec{w}_2}{\partial t} \cdot \vec{n}\, d\Gamma - \frac{1}{2\rho_2 c^2} \frac{d}{dt} \int\limits_{\Omega_2} |p_2|^2\, d\Omega_2 + \rho_2 \int\limits_{\Omega_2} \vec{f} \cdot \frac{\partial \vec{w}_2}{\partial t}\, d\Omega_2.$$

Adding the left and the right hand sides of these identities we receive the identity

$$\frac{d}{dt}\left\{\frac{1}{2}\rho_1 \int\limits_{\Omega_1} \left|\frac{\partial \vec{w}_1}{\partial t}\right|^2 d\Omega_1 + \frac{1}{2}\rho_2 \int\limits_{\Omega_2} \left|\frac{\partial \vec{w}_2}{\partial t}\right|^2 d\Omega_2 + \frac{1}{2\rho_2 c^2} \int\limits_{\Omega_2} |p_2|^2\, d\Omega_2\right\} +$$

$$+\int\limits_{\Gamma} (p_1 - p_2) \frac{\partial \zeta}{\partial t}\, d\Gamma = \sum_{k=1}^{2} \rho_k \int\limits_{\Omega_k} \vec{f} \cdot \frac{\partial \vec{w}_k}{\partial t}\, d\Omega_k. \tag{16}$$

We use now the First Green's Formula for Laplace – Beltrami operator (see, for instance, [23], p. 129, [24], p. 276):

$$-\int\limits_{\Gamma} \Delta_\Gamma u \cdot v\, d\Gamma = \int\limits_{\Gamma} \nabla_\Gamma u \cdot \nabla_\Gamma v\, d\Gamma - \int\limits_{\partial\Gamma} \frac{\partial u}{\partial e} v\, dS. \tag{17}$$

Then from (12) – (14) we have

$$\int\limits_{\Gamma} (p_1 - p_2) \frac{\partial \zeta}{\partial t}\, d\Gamma = \int\limits_{\Gamma} (\mathcal{L}_\sigma \zeta) \frac{\partial \zeta}{\partial t}\, d\Gamma = \frac{1}{2}\frac{d}{dt}(\zeta, \zeta)_{B_\sigma}, \tag{18}$$

where

$$(\zeta, \zeta)_{B_\sigma} := \int\limits_{\Gamma} \left[\sigma |\nabla_\Gamma \zeta|^2 + a_\sigma |\zeta|^2\right] d\Gamma + \sigma \oint\limits_{\partial\Gamma} \chi |\zeta|^2\, ds. \tag{19}$$

Therefore, it follows from (16) – (19) that the identity

$$\frac{1}{2}\sum_{k=1}^{2}\rho_k\int\limits_{\Omega_k}\left|\frac{\partial\vec{w}_k}{\partial t}(t,x)\right|^2 d\Omega_k+\frac{1}{2\rho_2c^2}\int\limits_{\Omega_2}|p_2(t,x)|^2\,d\Omega_2+\frac{1}{2}\left(\zeta(t,x),\zeta(t,x)\right)_{B_\sigma}=$$

$$=\frac{1}{2}\sum_{k=1}^{2}\rho_k\int\limits_{\Omega_k}\left|\vec{w_k}^1(x)\right|^2 d\Omega_k+\frac{1}{2\rho_2c^2}\int\limits_{\Omega_2}|p_2(0,x)|^2\,d\Omega_2+\frac{1}{2}\left(\zeta(0,x),\zeta(0,x)\right)_{B_\sigma}+$$

$$+\sum_{k=1}^{2}\rho_k\int\limits_0^t\left(\int\limits_{\Omega_k}\vec{f}(t,x)\cdot\frac{\partial\vec{w}_k}{\partial t}(t,x)\,d\Omega_k\right)dt \tag{20}$$

is valid. It is the law of full energy balance for considered hydrodynamical system. Note, that here

$$p_2(0,x)=-\rho_2c^2\mathrm{div}\,\vec{w_2}^0(x),\quad \zeta(0,x)=(\vec{w_1}^0(x)\cdot\vec{n})\,|_\Gamma=(\vec{w_2}^0(x)\cdot\vec{n})_\Gamma. \tag{21}$$

The first term from the left hand side of (20) is a kinetic energy of the system, the second and the fhird ones is a potential energy, consisting of the term for compressible gas and the term for free surface and acting gravity and surface tension on it. From the right hand side in (20) we have the sum of the full energy at the initial moment $t=0$ and the work of an external force $\vec{f}(t,x)$ on the interval $[0,t]$.

2.3. **Using the method of orthogonal projecting. Transition to the problem with scalar unknown function.** For investigation of problem (8) – (15) we use the method of orthogonal projecting (see, for instance, [9], Subsection 6.3.3). Introduce Hilbert spaces $\vec{L}_2(\Omega_i)$, $i=1,2$, with inner products

$$(\vec{u},\vec{v})_{\Omega_i}:=\int\limits_{\Omega_i}\vec{u}(x)\cdot\vec{v}(x)d\Omega_i \tag{22}$$

and corresponding norms. For the space $\vec{L}_2(\Omega_1)$ (ideal incompressible fluid) we take into consideration the following orthogonal decomposition (see [9], pp.117 – 118):

$$\vec{L}_2(\Omega_1)=\vec{J}_0(\Omega_1)\oplus\vec{G}_{0,\Gamma}(\Omega_1)\oplus\vec{G}_{h,S_1}(\Omega_1), \tag{23}$$

$$\vec{J}_0(\Omega_1):=\left\{\vec{v}\in\vec{L}_2(\Omega_1):\ \ \mathrm{div}\,\vec{v}=0\ \ (\text{in}\ \ \Omega_1),\ \ \vec{v}\cdot\vec{n}=0\ \ (\text{on}\ \ \partial\Omega_1)\right\}, \tag{24}$$

$$\vec{G}_{0,\Gamma}(\Omega_1):=\left\{\vec{u}\in\vec{L}_2(\Omega_1):\ \ \vec{u}=\nabla\varphi,\ \ \varphi=0\ \ (\text{on}\ \ \Gamma)\right\}, \tag{25}$$

$$\vec{G}_{h,S_1}(\Omega_1):=\left\{\vec{w}\in\vec{L}_2(\Omega_1):\ \ \vec{w}=\nabla\Phi,\ \ \Delta\Phi=0\ \ (\text{in}\ \ \Omega_1),\right.$$

$$\left.\frac{\partial\Phi}{\partial n}=0\ \ (\text{on}\ \ S_1),\ \ \int\limits_\Gamma\Phi\,d\Gamma=0\right\}. \tag{26}$$

It follows from (8) and (10) that if $\vec{w_1}(t,x)$ is a function in variable t with values from $\vec{L}_2(\Omega_1)$ then

$$\vec{w_1}(t,x) = \vec{v_1}(t,x) + \nabla\Phi_1(t,x) \in \vec{J}_0(\Omega_1) \oplus \vec{G}_{h,S_1}(\Omega_1), \tag{27}$$

$$\vec{v_1}(t,x) \in \vec{J}_0(\Omega_1), \quad \nabla\Phi_1(t,x) \in \vec{G}_{h,S_1}(\Omega_1). \tag{28}$$

If $\nabla p_1(t,x)$ is a function in t with values from $\vec{L}_2(\Omega_1)$ then

$$\nabla p_1(t,x) = \nabla\widetilde{p}_1(t,x) + \nabla\varphi_1(t,x) \in \vec{G}_{h,S_1}(\Omega_1) \oplus \vec{G}_{0,\Gamma}(\Omega_1), \tag{29}$$

$$\nabla\widetilde{p}_1(t,x) \in \vec{G}_{h,S_1}(\Omega_1), \quad \nabla\varphi_1(t,x) \in \vec{G}_{0,\Gamma}(\Omega_1). \tag{30}$$

Let $P_{1,0}$, $P_{1,0,\Gamma}$ and P_{1,h,S_1} be the orthoprojections on the subspaces (23), respectively. If we will use representations (27) and (29) in the first equation (8) and will act by these projections from the left, we will have relations

$$\rho_1\frac{\partial^2\vec{v}_1}{\partial t^2} = \rho_1 P_{1,0}\vec{f}, \quad \vec{v}_1(0,x) = P_{1,0}\vec{w}_1^0, \quad \frac{\partial\vec{v}_1}{\partial t}(0,x) = P_{1,0}\vec{w}_1^1; \tag{31}$$

$$\vec{0} + \nabla\varphi_1 = \rho_1 P_{1,0,\Gamma}\vec{f}; \tag{32}$$

$$\rho_1\frac{\partial^2}{\partial t^2}\nabla\Phi_1 + \nabla\widetilde{p}_1 = \rho_1 P_{1,h,S_1}\vec{f} =: \rho_1\nabla F_1. \tag{33}$$

It is evident that fields $\vec{v_1}$ and $\nabla\varphi_1$ can be finded immediately from (31) and (32). Therefore in further we must study only equation (33) and other equations and boundary conditions.

Consider now $\vec{L}_2(\Omega_2)$ and its decomposition

$$\vec{L}_2(\Omega_2) = \vec{G}(\Omega_2) \oplus \vec{J}_0(\Omega_2), \tag{34}$$

$$\vec{G}(\Omega_2) := \left\{ \vec{w} \in \vec{L}_2(\Omega_2) : \ \vec{w} = \nabla\Phi, \ \int\limits_{\Omega_2} \Phi \, d\Omega_2 = 0 \right\}, \tag{35}$$

$$\vec{J}_0(\Omega_2) := \Big\{ \vec{v} \in \vec{L}_2(\Omega_2) : \ \operatorname{div}\vec{v} = 0 \ \ (\text{in } \ \Omega_2), \ \ \vec{v}\cdot\vec{n} = 0 \ \ (\text{on } \ \partial\Omega_2) \Big\}. \tag{36}$$

(Here and in (23) – (26) operations $\operatorname{div}\vec{v}$ and $(\vec{v}\cdot\vec{n})_\Gamma$ are understood in sense of distributions, see, for instance, [9], pp. 111 – 114.)

If $\vec{w}_2(t,x)$ and $\nabla p_2(t,x)$ are functions in t with values in $\vec{L}_2(\Omega_2)$ then

$$\vec{w}_2(t,x) = \vec{v}_2(t,x) + \nabla\Phi_2(t,x), \tag{37}$$

$$\vec{v}_2(t,x) \in \vec{J}_0(\Omega_2), \ \ \nabla\Phi_2(t,x) \in \vec{G}(\Omega_2), \ \ \nabla p_2(t,x) \in \vec{G}(\Omega_2). \tag{38}$$

Indeed, it follows from the second equation (9) and from (11) that

$$\int\limits_{\Omega_2} p_2 \, d\Omega_2 = -\rho_2 c^2 \int\limits_{\Omega_2} \operatorname{div}\vec{w}_2 \, d\Omega_2 = -\rho_2 c^2 \int\limits_{\Gamma} \vec{w}_2\cdot\vec{n} \, d\Gamma = -\rho_2 c^2 \int\limits_{\Gamma} \zeta \, d\Gamma = 0. \tag{39}$$

Introduce the orthoprojections $P_{2,G}$ and $P_{2,0}$ on subspaces (34). Then, acting by these orthoprojections from the left in (9) and using (37), (38), we will have relations

$$\rho_2 \frac{\partial^2}{\partial t^2} \nabla \Phi_2 + \nabla p_2 = \rho_2 P_{2,G} \vec{f} =: \rho_2 \nabla F_2, \tag{40}$$

$$\rho_2 \frac{\partial^2 \vec{v}_2}{\partial t^2} = \rho_2 P_{2,0} \vec{f}, \quad \vec{v}_2(0,x) = P_{2,0} \vec{w}_2^0, \quad \frac{\partial}{\partial t} \vec{v}_2(0,x) = P_{2,0} \vec{w}_2^1. \tag{41}$$

It is evident that $\vec{v}_2(t,x)$ is defined uniquelly from problem (41), and therefore in further we must study equation (40) and others.

Let us transform boundary conditions (10) – (14) with taking into account (27), (29) and (26), (34), (35), (38). First of all, instead of (10) we have now conditions

$$\frac{\partial \Phi_1}{\partial n} = 0 \ \ (\text{on } \ S_1), \quad \frac{\partial \Phi_2}{\partial n} = 0 \ \ (\text{on } \ S_2), \tag{42}$$

and conditions (11) have the form

$$\frac{\partial \Phi_1}{\partial n} = \frac{\partial \Phi_2}{\partial n} =: \zeta \ \ (\text{on } \ \Gamma), \quad \int_\Gamma \zeta \, d\Gamma = 0. \tag{43}$$

Further, it follows from (33) and (40) that

$$\rho_i \frac{\partial^2 \Phi_i}{\partial t^2} + p_i = \rho_i F_i + c_i(t) \quad (\text{in } \ \Omega_i, \quad i = 1, 2), \tag{44}$$

where $c_i(t)$ are arbitrary functions in t. But from (39) and corresponding conditions for Φ_2 and F_2 (see (35)) we conclude, that $c_2(t) \equiv 0$. Then condition (12) can be rewritten in the form

$$\rho_1 \frac{\partial^2 \Phi_1}{\partial t^2} - \rho_2 \frac{\partial^2 \Phi_2}{\partial t^2} + \mathcal{L}_\sigma \zeta = \rho_1 F_1 - \rho_2 F_2 + c_1(t) \quad (\text{on } \ \Gamma). \tag{45}$$

Introduce Hilbert space $L_2(\Gamma)$ with ordinary scalar product

$$(\zeta, \eta)_0 := \int_\Gamma \zeta(x) \eta(x) \, d\Gamma. \tag{46}$$

Then the last condition (11) can be written as

$$\int_\Gamma \zeta \, d\Gamma = (\zeta, 1_\Gamma)_0 = 0, \tag{47}$$

where 1_Γ is a unique function defined on Γ. It means that

$$\zeta \in L_{2,\Gamma} := L_2(\Gamma) \ominus \{1_\Gamma\}. \tag{48}$$

Let P_Γ be the orthoprojection from $L_2(\Gamma)$ onto $L_{2,\Gamma}$ that is

$$P_\Gamma \eta := \eta - |\Gamma|^{-1} \int_\Gamma \eta \, d\Gamma, \quad \forall \eta \in L_2. \tag{49}$$

Then $P_\Gamma \zeta = \zeta$ (see (47)), and acting by the operator P_Γ from the left in (45), we will have the condition

$$\rho_1 \frac{\partial^2 \Phi_1}{\partial t^2} - \rho_2 \frac{\partial^2}{\partial t^2} (P_\Gamma \Phi_2) + B_\sigma \zeta = \rho_1 F_1 - \rho_2 P_\Gamma F_2 \quad (\text{on } \Gamma). \tag{50}$$

We took into account in (50) that conditions

$$\int_\Gamma \Phi_1 \, d\Gamma = \int_\Gamma F_1 \, d\Gamma = 0$$

hold, see (26). By definition, the operator B_σ is defined by the law

$$B_\sigma := P_\Gamma \mathcal{L}_\sigma P_\Gamma, \quad \mathcal{D}(B_\sigma) = \mathcal{D}(\mathcal{L}_\sigma) \subset L_{2,\Gamma}. \tag{51}$$

Lemma 1. *The operator B_σ with the domain*

$$\mathcal{D}(B_\sigma) := \left\{ \zeta \in H^2(\Gamma) \cap L_{2,\Gamma} : \ \frac{\partial \zeta}{\partial e} + \chi \zeta = 0 \ \ (on \ \partial\Gamma) \right\} \tag{52}$$

is bounded from below self–adjoint operator acting in the space $L_{2,\Gamma}$. Its quadratic form (see (19)*) is*

$$(B_\sigma \zeta, \zeta)_0 = (\zeta, \zeta)_{B_\sigma} = \int_\Gamma [\sigma \, |\nabla_\Gamma \zeta|^2 + a \, |\zeta|^2] d\Gamma + \oint_{\partial\Gamma} \chi \, |\zeta|^2 \, d\Gamma \tag{53}$$

and there exists $\gamma \in \mathbb{R}$ such that

$$(\zeta, \zeta)_{B_\sigma} \geqslant \gamma \, \|\zeta\|_0^2, \quad \forall \, \zeta \in \mathcal{D}(B_\sigma). \tag{54}$$

These properties are valid for sufficiently smooth $\partial\Gamma$.

Proof. of the lemma is done in [9], p. 205. □

Transform now the second condition in (9). By (44), (35),(38),

$$p_2 = -\rho_2 \frac{\partial^2 \Phi_2}{\partial t^2} + \rho_2 F_2, \quad (c_2(t) \equiv 0), \quad \operatorname{div} \vec{w}_2 = \Delta \Phi_2.$$

Therefore for unknown function $\Phi_2(t,x)$ we have the equation

$$\frac{\partial^2 \Phi_2}{\partial t^2} = c^2 \Delta \Phi_2 + F_2 \quad (\text{in } \Omega_2),$$

and unknown function $\Phi_1(t,x)$ is a harmonic one:

$$\Delta \Phi_1 = 0 \quad (\text{in } \Omega_1).$$

We can now formulate the statement of the initial boundary value problem for unknown scalar function $\Phi_i(t,x)$, $i = 1,2$:

$$\Delta \Phi_1 = 0 \quad (\text{in } \Omega_1), \tag{55}$$

$$\frac{\partial^2 \Phi_2}{\partial t^2} = c^2 \Delta \Phi_2 + F_2(t,x) \quad (\text{in } \Omega_2), \tag{56}$$

$$\frac{\partial \Phi_1}{\partial n} = 0 \quad (\text{on } S_1), \quad \frac{\partial \Phi_2}{\partial n} = 0 \quad (\text{on } S_2), \tag{57}$$

$$\frac{\partial \Phi_1}{\partial n} = \frac{\partial \Phi_2}{\partial n} =: \zeta \quad (\text{on } \Gamma), \quad \int_\Gamma \zeta d\Gamma = 0, \tag{58}$$

$$\rho_1 \frac{\partial^2 \Phi_1}{\partial t^2} - \rho_2 \frac{\partial^2}{\partial t^2}(P_\Gamma \Phi_2) + B_\sigma \zeta = \rho_1 F_1 - \rho_2 P_\Gamma F_2 \quad (\text{on } \Gamma), \tag{59}$$

$$\nabla \Phi_1(0,x) = \nabla \Phi_1^0(x) = P_{h,S_1} \vec{w}_1^0(x), \quad \nabla \Phi_2(0,x) = \nabla \Phi_2^0(x) = P_G \vec{w}_2^0(x), \tag{60}$$

$$\frac{\partial}{\partial t} \nabla \Phi_1(0,x) = \nabla \Phi_1^1(x) = P_{h,S_1} \vec{w}_1^1(x), \quad \frac{\partial}{\partial t} \nabla \Phi_2(0,x) = \nabla \Phi_2^1(x) = P_G \vec{w}_2^1(x). \tag{61}$$

Initial boundary value problem (55) – (61) has the following peculiarity: the second derivatives with respect to t are located both in equation (56) and in boundary condition (59).

2.4. **The problem on eigenoscillations.** Consider eigenoscillations of the hydrosystem „fluid - gas", that is, solutions to the homogeneous problem (55) – (61) such that its change in t by the law $\exp(i\omega t)$ where ω is a frequency of eigenoscillations.

If $\vec{f}(t,x) \equiv \vec{0}$, then $F_1(t,x) \equiv 0$, $F_2(t,x) \equiv 0$. We set

$$\Phi_i(t,x) = \exp(i\omega t) \Phi_i(x), \quad i = 1, 2, \tag{62}$$

where $\Phi_i(x)$ are so called amplitude functions. From (55) – (61) we have the following spectral problem for these functions:

$$\Delta \Phi_1 = 0 \ (\text{in } \Omega_1), \tag{63}$$

$$-\Delta \Phi_2 = \lambda c^{-2} \Phi_2 \ (\text{in } \Omega_2), \quad \lambda := \omega^2, \tag{64}$$

$$\frac{\partial \Phi_1}{\partial n} = 0 \ (\text{on } S_1), \quad \frac{\partial \Phi_2}{\partial n} = 0 \ (\text{on } S_2), \tag{65}$$

$$\frac{\partial \Phi_1}{\partial n} = \frac{\partial \Phi_2}{\partial n} =: \zeta \ (\text{on } \Gamma), \tag{66}$$

$$B_\sigma \zeta = \lambda \left(\rho_1 \Phi_1 - \rho_2 P_\Gamma \Phi_2 \right) \ (\text{on } \Gamma). \tag{67}$$

$$\int_\Gamma \zeta \, d\Gamma = 0, \quad \lambda \int_{\Omega_2} \Phi_2 \, d\Omega_2 = 0. \tag{68}$$

Here λ is a spectral parameter of the problem, $\Phi_1(x)$ and $\Phi_2(x)$ are unknown amplitude functions. We see that spectral parameter λ enters as in equation (64) as in boundary condition (67). The operator B_σ is defined by (51), (52) and has properties (53), (54) (see Lemma 1). The last relation in (68) is a corollary of equations (64) – (66) and the first relation (68):

$$\int_{\Gamma_2} (-\Delta \Phi_2) \, d\Omega_2 = \lambda c^{-2} \int_{\Omega_2} \Phi_2 \, d\Omega_2 = \int_{\Omega_2} \nabla \Phi_2 \cdot \nabla 1 \, d\Omega_2 - \int_{\partial \Omega_2} \frac{\partial \Phi_2}{\partial n_2} \cdot 1 \, dS =$$

$$= \int_\Gamma \frac{\partial \Phi_2}{\partial n} \, d\Gamma = \int_\Gamma \zeta \, d\Gamma = 0.$$

Definition 1. We say that the investigated hydrosystem is statically stable in linear approximation if the operator B_σ is positive definite ($B_\sigma \gg 0$), that is,

$$(B_\sigma \zeta, \zeta)_0 \geqslant c \, \|\zeta\|_0^2, \quad c > 0, \quad \zeta \in \mathcal{D}(B_\sigma). \qquad \square \tag{69}$$

If inequality (69) holds then one can introduce the energetic space H_{B_σ} of the operator B_σ (see, for instance, [25]), i.e., the set of elements $\zeta \in L_{2,\Gamma}$ such that the norm $\|\zeta\|_{B_\sigma}^2 < \infty$.

Lemma 2. *The energetic norm*

$$\|\zeta\|_{B_\sigma}^2 := \int_\Gamma [\sigma \, |\nabla_\Gamma \zeta|^2 + a \, |\zeta|^2] d\Gamma + \oint_{\partial\Gamma} \chi \, |\zeta|^2 \, ds$$

is equivalent to the norm

$$\|\zeta\|_\nabla^2 := \int_\Gamma |\nabla_\Gamma \zeta|^2 \, d\Gamma, \qquad \int_\Gamma \zeta \, d\Gamma = 0,$$

and this norm is equivalent to standart norm

$$\|\zeta\|_{1,\Gamma}^2 := \int_\Gamma [|\nabla_\Gamma \zeta|^2 + |\zeta|^2] d\Gamma$$

of the space $H^1(\Gamma)$.

Proof. See [9], p.206. $\square$

It follows from Lemma 2 and embedding theorem ($H^1(\Gamma)$ is compact embedded in $L_2(\Gamma)$) that the operator $B_\sigma (\gg 0)$ has a discrete positive spectrum consisting of finite-multiple eigenvalues $\{\lambda_k(B_\sigma)\}_{k=1}^\infty$ with limit point $\lambda = +\infty$. The system of eigenelements of the operator B_σ forms an orthogonal basis in $L_{2,\Gamma}$ and $H_{B_\sigma} = H^1(\Gamma) \cap L_{2,\Gamma} = \mathcal{D}(B_\sigma^{1/2})$. Further, the inverse operator B_σ^{-1} is compact and positive in the space $L_{2,\Gamma}$.

Let's derive preliminary some simple properties of solutions to spectral problem (63) – (68).

If condition (69) holds then we can find engenvalue λ, corresponding to solution $\{\Phi_1(x), \Phi_2(x)\}$, calculating the values of the functional

$$F_1(\Phi_1; \Phi_2) := \frac{\sum_{k=1}^{2} \rho_k \int_{\Omega_k} |\nabla \Phi_k|^2 \, d\Omega_k}{\rho_2 c^{-2} \int_{\Omega_2} | \, \Phi_2 \, |^2 \, d\Omega_2 + \left\| B_\sigma^{-1/2} P_\Gamma (\rho_1 \Phi_1 - \rho_2 \Phi_2) \right\|_0^2}. \tag{70}$$

It follows from (70) that $\lambda = F_1(\Phi_1; \Phi_2) > 0$, that is, frequencies of oscillations $\omega = \pm\sqrt{\lambda}$ are real numbers. This fact is evident physically because the investigated hydrosystem is conservative (not dissipative).

Functional (70) can be find for solutions of spectral problem (63) – (68) by the following way. We multiply the both part of equations (63), (64) on $\rho_i\Phi_i$ and integrate over Ω_i; further we use the First Green's Formula for Laplace operator, boundary conditions (65) – (66) and summize these identities. Since the operator B_σ is positive definite ($B_\sigma \gg 0$) then it has positive inverse operator B_σ^{-1} and condition (67) can be rewritten in the form

$$\zeta = \lambda B_\sigma^{-1} P_\Gamma(\rho_1\Phi_1 - \rho_2\Phi_2) \quad (\text{on } \Gamma). \tag{71}$$

(Remind that $\int_\Gamma \Phi_1\,d\Gamma = 0$ and therefore $P_\Gamma\Phi_1 = \Phi_1$, see (26).) Therefore

$$(B_\sigma^{-1} P_\Gamma(\rho_1\Phi_1 - \rho_2\Phi_2), (\rho_1\Phi_1 - \rho_2\Phi_2))_0 = \tag{72}$$
$$= ((B_\sigma^{-1} P_\Gamma(\rho_1\Phi_1 - \rho_2\Phi_2), P_\Gamma(\rho_1\Phi_1 - \rho_2\Phi_2))_0 = \left\| B_\sigma^{-1/2} P_\Gamma(\rho_1\Phi_1 - \rho_2\Phi_2)\right\|_0^2.$$

3. Operator approach to investigation of the spectral problem.

In this section, for investigation of spectral problem (63) – (68) we use an operator approach which is based on introduction of auxiliary boundary value problems and its operators and on transition from (63) – (68) to the spectral problem for some operator equation in Hilbert space.

3.1. **Auxiliary boundary value problems.** Consider auxiliary boundary value problems directly connected with spectral problem (63) – (68).

We introduce preliminary the following necessary in further Hilbert spaces of scalar functions.

1^0. The spaces $L_2(\Omega_i)$ with inner product

$$(u, v)_{\Omega_i} := \int_{\Omega_i} u(x)\overline{v(x)}\,d\Omega_i, \quad i = 1, 2. \tag{73}$$

2^0. The space $L_2(\Gamma)$ with inner products

$$(\varphi, \psi)_0 := \int_\Gamma \varphi(x)\overline{\psi(x)}\,d\Gamma. \tag{74}$$

3^0. The space $H^1(\Omega_1)$ with the norm

$$\|u\|_{1,\Omega_1}^2 := \int_{\Omega_1} |\nabla u|^2\, d\Omega_1 + \left| \int_\Gamma u\,d\Gamma \right|^2, \tag{75}$$

that is equivalent to the standart norm of Sobolev space $W_2^1(\Omega_1)$.

4^0. The space $H^1(\Omega_2)$ with the norm

$$\|u\|_{1,\Omega_2}^2 := \int_{\Omega_2} |\nabla u|^2\, d\Omega_2 + \left| \int_{\Omega_2} u\,d\Omega_2 \right|^2, \tag{76}$$

that is equivalent to the standart norm of Sobolev space $W_2^1(\Omega_2)$.

5^0. The space

$$H = H_0 := L_2(\Gamma) \ominus \{1_\Gamma\} = L_{2,\Gamma}, \tag{77}$$

where 1_Γ is the function that is equal to 1 on Γ. We consider also the equipment (see [26], Section 1.1; and also [9])

$$H_+ \subset H_0 \subset H_-, \tag{78}$$

where

$$H_+ = W_2^{1/2}(\Gamma) \cap H_0, \quad H_0 = (H_+)^*, \tag{79}$$

that is H_- is a dual space to H_+ (in inner product of the space H_0). Namely, if $u \in H_+$ and $v \in H_-$, then linear bounded functional $l_v(u)$ has the norm $l_v(u) := \langle u, v\rangle_0$ and

$$|l_v(u)| \leqslant \|u\|_+ \cdot \|v\|_- \,. \tag{80}$$

Here $\langle u, v\rangle_0$ is an extention by continuity of the inner product $(u, v)_0$ on the case when $u \in H_+$, $v \in H_-$.

In this paper, we will consider that regions Ω_1 and Ω_2 are lipshitsian domains, in particularly, its can be piecewise smooth domains with nonzero inner and outer dihedral angles between smooth parts of $\partial\Omega_i$, $i = 1, 2$.

We will denote by $H^1_{\Omega_i} \subset H^1(\Omega_i)$, $i = 1, 2$, the subspaces of spaces with norms (75) and (76) such that the conditions

$$\int_\Gamma u\, d\Gamma = 0, \quad \int_{\Omega_2} u\, d\Omega_2 = 0 \tag{81}$$

are valid for elements of $H^1(\Omega_1)$ and $H^1(\Omega_2)$, respectively. Then, by (75) and (76), we will have

$$\|u\|^2_{1,\Omega_i} = \int_{\Omega_i} |\nabla u|^2 \, d\Omega_i, \quad i = 1, 2, \quad u \in H^1_{\Omega_i}, \tag{82}$$

that is, squared norms are equal to Dirichlet integral.

Consider, on the base of introduced spaces, the following auxiliary boundary value problems.

Problem 1. For known function $\zeta(x)$, $x \in \Gamma$, find generalized solution $\Phi_1(x)$ to the problem

$$\Delta\Phi_1(x) = 0 \text{ (in } \Omega_1), \quad \frac{\partial\Phi_1}{\partial n} = 0 \text{ (on } S_1), \tag{83}$$

$$\frac{\partial\Phi_1}{\partial n} = \zeta \text{ (on } \Gamma), \quad \int_\Gamma \zeta\, d\Gamma = 0, \quad \int_\Gamma \Phi_1\, d\Gamma = 0. \qquad \square$$

Definition 2. A function $\Phi_1(x) \in H^1_{\Omega_1}$ is said to be a weak solution to Problem 1 if the identity

$$(\Psi, \Phi_1)_{1,\Omega_1} = \langle \gamma_1 \Psi, \zeta \rangle_0 \tag{84}$$

is valid for any $\Psi \in H^1_{\Omega_1}$. Here $\gamma_1 : H^1_{\Omega_1} \longrightarrow H_0$ is a trace operator, i.e.,

$$\gamma_1 \left(\Psi \mid_{\Omega_1} \right) := \Psi \mid_\Gamma . \qquad \square \tag{85}$$

It follows from the First Green's Formula for Laplace operator (and domain Ω_1) that a classical solution to Problem 1 is a weak one.

It is known (see, for instance, [9], pp. 105 – 106) that Problem 1 has a unique weak solution $\Phi_1 \in H^1_{\Omega_1}$, $\Phi_1 = T_1\zeta$, if and only if

$$\zeta \in H_- = (H_+)^* = \{\zeta \in W_2^{-1/2}(\Gamma) : \int_\Gamma \zeta \, d\Gamma = 0\}. \tag{86}$$

Here $T_1 : H_- \longrightarrow H^1_{\Omega_1}$ is a linear bounded operator with bounded inverse on the image $\mathcal{R}(T_1) \subset H^1_{\Omega_1}$ of the operator T_1.

Problem 2. For known function $\zeta(x)$, $x \in \Gamma$, find weak solution $\Phi_{22}(x)$ to the problem

$$\Delta\Phi_{22} = 0 \text{ (in } \Omega_2), \quad \frac{\partial \Phi_{22}}{\partial n} = 0 \text{ (on } S_2), \tag{87}$$

$$\frac{\partial \Phi_{22}}{\partial n} = -\zeta \text{ (on } \Gamma), \quad \int_\Gamma \zeta \, d\Gamma = 0, \quad \int_{\Omega_2} \Phi_{22} \, d\Omega_2 = 0. \qquad \square$$

Definition 3. A function $\Phi_{22}(x) \in H^1_{\Omega_2}$ is said to be a weak solution to Problem 2 if the identity

$$(\Psi, \Phi_{22})_{1,\Omega_2} = -\langle \gamma_2 \Psi, \zeta \rangle_0 \tag{88}$$

is valid for any $\Psi \in H^1_{\Omega_2}$. Here $\gamma_2 : H^1_{\Omega_2} \longrightarrow H_0$ is a trace operator. $\square$

It follows from the First Green's Formula for Laplace operator (and domain Ω_2) that a classical solution to Problem 2 is a weak one.

Problem 2 (as Problem 1) has a unique weak solution $\Phi_{22} \in H^1_{\Omega_2}$ if and only if condition (86) is valid (see once more, for instance, [9], pp. 105 – 106). Then

$$\Phi_{22} = T_2\zeta, \quad T_2 : H_- \longrightarrow H^1_{\Omega_2}, \tag{89}$$

T_2 is a bounded linear operator with bounded inverse on the image $\mathcal{R}(T_2) \subset H^1_{\Omega_2}$.

Problem 3. For known function $f(x)$, $x \in \Omega_2$, find weak solution $\Phi_{21}(x)$ to the problem

$$-\Delta\Phi_{21} = f \text{ (in } \Omega_2), \quad \frac{\partial \Phi_{21}}{\partial n} = 0 \text{ (on } S_2), \quad \frac{\partial \Phi_{21}}{\partial n} = 0 \text{ (on } \Gamma), \tag{90}$$

$$\int_{\Omega_2} f \, d\Omega_2 = 0, \quad \int_{\Omega_2} \Phi_{21} \, d\Omega_2 = 0. \qquad \square$$

Definition 4. A function $\Phi_{21} \in H^1_{\Omega_2}$ is said to be a weak solution to Problem 3 if the identity

$$(\Psi, \Phi_{21})_{1,\Omega_2} = \langle \Psi, f \rangle_{\Omega_2} \tag{91}$$

holds for an $\Psi \in H^1_{\Omega_2}$. □

By $\langle u, v \rangle_{\Omega_2}$ we denote here the linear bounded functional for $u \in H^1_{\Omega_2}$ and $v \in (H^1_{\Omega_2})^*$. We use here the equipment

$$H^1_{\Omega_2} \subset L_{2,\Omega_2} \subset (H^1_{\Omega_2})^*, \quad L_{2,\Omega_2} = L_2(\Omega_2) \ominus \{1_{\Omega_2}\}. \tag{92}$$

It follows from the First Green's Formula for laplace operator (and domain Ω_2) that a classical solution to Problem 3 is a weak one.

Problem 3 has a unique weak solution $\Phi_{21} \in H^1_{\Omega_2}$ if and only if (see [9], pp. 97)

$$f(x) \in (H^1_{\Omega_2})^*. \tag{93}$$

Then

$$\Phi_{21} = A^{-1} f, \quad A^{-1} : (H^1_{\Omega_2})^* \longrightarrow H^1_{\Omega_2}, \quad A : H^1_{\Omega_2} \longrightarrow (H^1_{\Omega_2})^*. \tag{94}$$

It is known (see, for instance, [9]), that the restriction of A, such that $\mathcal{R}(A) = L_{2,\Omega_2}$, is a selfadjoint positive definite operator with compact inverse operator, i.e., $A^{-1} : L_{2,\Omega_2} \longrightarrow L_{2,\Omega_2}$, $A^{-1} \in \mathfrak{S}_\infty(L_{2,\Omega_2})$. The operator $A : \mathcal{D}(A) \subset L_{2,\Omega_2} \longrightarrow L_{2,\Omega_2}$ has a discrete spectrum $\{\lambda_k(A)\}_{k=1}^\infty \subset \mathbb{R}_+$ and

$$\lambda_k(A) = \left(\frac{\mid \Omega_2 \mid}{6\pi^2} \right)^{-2/3} k^{2/3} [1 + o(1)], \quad k \longrightarrow \infty, \quad \Omega_2 \subset \mathbb{R}^3. \tag{95}$$

From this it follows that the operator A^{-1} belongs to the class of compact operators $\mathfrak{S}_p$ for $p > 3/2$. We have also the properties

$$\mathcal{D}(A) \subset H^1_{\Omega_2}, \quad \mathcal{D}(A^{1/2}) = H^1_{\Omega_2}, \quad A : \mathcal{D}(A) \subset L_{2,\Omega_2} \longrightarrow L_{2,\Omega_2}. \tag{96}$$

3.2. **Transition to the operator problem in some Hilbert space.** Consider spectral problem (63) – (68) and suppose that $\Phi_1(x)$ is a weak solution to auxiliary Problem 1. Then

$$\Phi_1 \mid_{\Omega_1} = T_1 \zeta, \quad \gamma_1 \Phi_1 = \gamma_1 T_1 \zeta =: C_1 \zeta. \tag{97}$$

We represent $\Phi_2(x)$ in the form

$$\Phi_2(x) = \Phi_{21}(x) + \Phi_{22}(x), \tag{98}$$

where $\Phi_{22}(x)$ is a weak solution to auxiliary Problem 2 and $\Phi_{21}(x)$ is a weak solution to auxiliary Problem 3 for $f = \lambda c^{-2} \Phi_2$. Then

$$\Phi_{21} \mid_{\Omega_2} = A^{-1}(\lambda c^{-2} \Phi_2), \quad \Phi_{22} \mid_{\Omega_2} = T_2 \zeta, \quad \gamma_2 \Phi_{22} = \gamma_2 T_2 \zeta =: -C_2 \zeta. \tag{99}$$

For simplicity we denote

$$\Phi_{21} \mid_{\Omega_2} =: \eta(x). \tag{100}$$

With account of (97) – (100) we can rewrite equations and boundary conditions (63) – (68) in the form

$$A\eta = \lambda c^{-2}\left(\eta + T_2\zeta\right), \quad \eta \in \mathcal{D}(A), \tag{101}$$

$$B_\sigma\zeta = \lambda\left(-\rho_2 P_\Gamma \gamma_2 \eta + P_\Gamma(\rho_1 C_1 + \rho_2 C_2)\right), \quad \zeta \in \mathcal{D}(B_\sigma). \tag{102}$$

Introduce the Hilbert space

$$\mathcal{H}(\Omega) := L_{2,\Omega_2} \oplus H_0, \quad L_{2,\Omega_2} := L_2(\Omega_2) \ominus \{1_{\Omega_2}\}, \tag{103}$$

for elements of the form $z = (\eta;\zeta)^t$ (by symbol $(\cdot\,;\cdot)^t$ we denote the operation of transforming) with the norm

$$\|z\|^2_{\mathcal{H}} := \|\eta\|^2_{L_{2,\Omega_2}} + \|\zeta\|^2_0. \tag{104}$$

We will consider that

$$\eta \in \mathcal{D}(A) \subset H^1_{\Omega_2} \subset L_{2,\Omega_2}, \tag{105}$$

and introduce new anknown elements

$$\psi := c\sqrt{\rho_2}A\eta, \quad \varphi := B_\sigma^{1/2}\zeta, \tag{106}$$

in (101). Then instead of (101) we will have a spectral problem

$$y = \lambda\mathcal{A}y, \quad y \in \mathcal{H}(\Omega), \tag{107}$$

where

$$\mathcal{A} := \begin{pmatrix} c^{-2}A^{-1} & \rho_2^{1/2}c^{-1}A^{-1/2}(A^{1/2}T_2)B_\sigma^{-1/2} \\ -\rho_2^{1/2}c^{-1}B_\sigma^{-1/2}P_\Gamma(\gamma_2 A^{-1/2})A^{-1/2} & B_\sigma^{-1/2}CB_\sigma^{-1/2} \end{pmatrix}, \tag{108}$$

$$C := P_\Gamma(\rho_1 C_1 + \rho_2 C_2)P_\Gamma, \quad y := (\psi;\varphi)^t. \tag{109}$$

These transforms show us that an initial spectral problem (63) – (68) is equivalent to problem (107) – (109) on finding of characteristic numbers λ and eigenelements y for the operator matrix $\mathcal{A}$ that acting in orthogonal sum of Hilbert spaces (103).

3.3. **The solutions properties of spectral problem.** Before investigation of solutions properties of problem (107) we will study properties of the operator matrix $\mathcal{A}$ from (108).

It will be shown that all elements of the matrix $\mathcal{A}$ are not only bounded but compact operators also and therefore $\mathcal{D}(\mathcal{A}) = \mathcal{H}(\Omega)$.

Introduce in the space H_- (see (78), (79)) the norm in one of equivalent forms (see, for instance, [9], pp. 101-103):

$$\|\zeta\|^2_{H_-} := \rho_1\int_{\Omega_1}|\nabla\Phi_1|^2\,d\Omega_1 + \rho_2\int_{\Omega_2}|\nabla\Phi_{22}|^2\,d\Omega_2, \tag{110}$$

where Φ_1 and Φ_{22} are generalized solutions to auxiliary problems 1 and 2.

Lemma 3. *The operator $C = P_\Gamma(\rho_1 C_1 + \rho_2 C_2)P_\Gamma : H_0 \longrightarrow H_0$ is a positive and compact operator. Its extension on $H_- \supset H_0$ is an isometric operator mapping H_- onto H_+. In this, $\mathcal{D}(C^{-1/2}) = H_+$, and after extention $\mathcal{D}(C^{-1/2}) = H_0$, $\mathcal{R}(C^{-1/2}) = H_-$.*

Proof. of the lemma see in [9], pp. 193 – 194. □

Lemma 4. *The operator $A^{1/2}T_2 : H_0 \longrightarrow L_{2,\Omega_2}$ and $-P_\Gamma(\gamma_2 A^{-1/2}) : L_{2,\Omega_2} \longrightarrow H_0$ are mutual adjoint compact operator.*

Proof. We will use the identities (88) and (96). Namely, it follows form (88) that

$$(\Psi, \Phi_{22})_{1,\Omega_2} = \left(A^{1/2}\Psi, A^{1/2}\Phi_{22}\right)_{\Omega_2} = -\langle \gamma_2\Psi, \zeta\rangle_0.$$

Since $\Phi_{22} = T_2\zeta$ then after substitution $A^{1/2}\Psi = v$ we receive from this (for $\zeta \in H_0$ we have $\langle \gamma_2\Psi, \zeta\rangle_0 = (\gamma_2\Psi, \zeta)_0$) the relation

$$\begin{aligned}\left(A^{1/2}T_2\zeta, v\right)_{\Omega_2} &= -\left(\zeta, \gamma_2 A_2^{-1/2}v\right)_0 = -\left(P_\Gamma\zeta, \gamma_2 A^{-1/2}v\right)_0 = \\ &= \left(\zeta, -P_\Gamma(\gamma_2 A^{-1/2}v)\right)_0, \quad \forall\, v \in H_0, \quad \forall\, v \in L_{2,\Omega_2}.\end{aligned} \tag{111}$$

Here we used the fact that the operator P_Γ introduced according to the law (49) is an orthoprojection, i.e.,

$$P_\Gamma = P_\Gamma^2 = P_\Gamma^* : L_2(\Gamma) \longrightarrow H_0 = H = L_2(\Gamma) \ominus \{1_\Gamma\}\,. \tag{112}$$

It follows from (111) that $(A^{1/2}T_2)^* = -P_\Gamma(\gamma_2 A^{-1/2})$ and both of these operators are bounded. But the operator $\gamma_2 A^{-1/2} : L_{2,\Omega_2} \longrightarrow L_2(\Gamma)$ is compact. Indeed, the operator $A^{-1/2} : L_{2,\Omega_2} \longrightarrow H^1_{\Omega_2}$ is bounded and the trace operator $\gamma_2 : H^1_{\Omega_2} \longrightarrow L_2(\Gamma)$ (by trace theorem of Gagliargo, see [27]) is compact. More precisely, γ_2 is bounded from $H^1_{\Omega_2}$ onto the space $H_+ = W_2^{1/2}(\Gamma) \cap H_0$ and H_+ is compact embedded into H_0. □

As a corollary of Lemmas 3 and 4 we have the following assertion.

Lemma 5. *Matrix operator $\mathcal{A}$ from* (108) *is a compact operator acting in $\mathcal{H}(\Omega)$.*

Proof. Remind that we used inequality (69) and therefore the operator B_σ has a bounded inverse operator $B_\sigma^{-1/2}$, acting in H_0. Therefore all entries in (108) are compact operators since A^{-1}, $A^{-1/2}$, $A^{1/2}T_2$, $-P_\Gamma(\gamma_2)A^{-1/2}$ and C are compact operators, and $B_\sigma^{-1/2}$ is bounded. □

Theorem 1. *Matrix operator $\mathcal{A}$ is positive selfadjoint compact operator acting in $\mathcal{H}(\Omega)$.*

Proof. By Lemmas 3 – 4, it is sufficient to check the property of positiveness of the operator $\mathcal{A}$.

For an arbitrary element $y = (\psi; \varphi)^t \in \mathcal{H}(\Omega)$ we consider he quadratic form of the operator $\mathcal{A}$. We have

$$(\mathcal{A}y, y)_{\mathcal{H}(\Omega)} = c^{-1}(A^{-1}\psi, \psi)_{\Omega_2} + \rho_2^{1/2}c^{-1}(T_2 B_\sigma^{-1/2}\varphi, \psi)_{\Omega_2} -$$

$$-\rho_2^{1/2}c^{-1}(B_\sigma^{-1/2}P_\Gamma\gamma_2A^{-1}\psi,\varphi)_0+(CB_\sigma^{-1/2}\varphi,B_\sigma^{-1/2}\varphi)_0. \tag{113}$$

Taking into account substitutions (106) we have from (113)

$$(\mathcal{A}y,y)_{\mathcal{H}(\Omega)}=\rho_2(\eta,A\eta)_{\Omega_2}+\rho_2(T_2\xi,A\eta)_{\Omega_2}-\rho_2(P_\Gamma\gamma_2\eta,\zeta)_0+(C\zeta,\zeta)_0. \tag{114}$$

Here

$$(C\zeta,\zeta)_0=\rho_1(C_1\zeta,\zeta)_0+\rho_2(C_2\zeta,\zeta)_0=\rho_1(\gamma_1T_1\zeta,\zeta)_0-\rho_2(\gamma_2T_2\zeta,\zeta)_0. \tag{115}$$

From this, using identities (84), (88) and denotations (97), (99), we have

$$(\gamma_1T_1\zeta,\zeta)_0=(\gamma_1\Phi_1,\zeta)_0=(\Phi_1,\Phi_1)_{1,\Omega_1}=\int\limits_{\Omega_1}|\nabla\Phi_1|^2\,d\Omega_1, \tag{116}$$

$$-(\gamma_2T_2\zeta,\zeta)_0=\int\limits_{\Omega_2}|\nabla\Phi_{22}|^2\,d\Omega_2. \tag{117}$$

Analogous considerations give us equalities

$$(\eta,A\eta)_{\Omega_2}=(A^{1/2}\eta,A^{1/2}\eta)_{\Omega_2}=\|\Phi_{21}\|^2_{1,\Omega_2}=\int\limits_{\Omega_2}|\nabla\Phi_{21}|^2\,d\Omega_2, \tag{118}$$

$$(T_2\zeta,A\eta)_{\Omega_2}=(A^{1/2}T_2\zeta,A^{1/2}\eta)_{\Omega_2}=(\Phi_{22},\Phi_{21})_{1,\Omega_2}=\int\limits_{\Omega_2}\nabla\Phi_{22}\cdot\overline{\nabla\Phi_{21}}\,d\Omega_2, \tag{119}$$

$$-(P_\Gamma\gamma_2\eta,\zeta)_0=-(\gamma_2\eta,\zeta)_0=(\Phi_{21},\Phi_{22})_{1,\Omega_2}=\int\limits_{\Omega_2}\nabla\Phi_{21}\cdot\overline{\nabla\Phi_{22}}\,d\Omega_2. \tag{120}$$

It follows from (114) – (120) that

$$(\mathcal{A}y,y)_{\mathcal{H}(\Omega)}=\rho_1\int\limits_{\Omega_1}|\nabla\Phi_1|^2\,d\Omega_1+\rho_2\int\limits_{\Omega_2}|\nabla\Phi_2|^2\,d\Omega_2\geqslant 0, \tag{121}$$

where $\Phi_2=\Phi_{21}+\Phi_{22}$. Consequently, the operator $\mathcal{A}=\mathcal{A}^*\geqslant 0$. If $(\mathcal{A}y,y)_{\mathcal{H}(\Omega)}=0$, then from (121) we have $\Phi_1(x)\equiv c_1=0$, $\Phi_2(x)\equiv c_2=0$, and therefore the operator $\mathcal{A}$ is positive. $\square$

It follows from above that spectral problem (107) is equivalent to the eigenvalue problem for compact positive operator $\mathcal{A}$, i.e.,

$$\mathcal{A}y=\mu y,\quad \mu=\lambda^{-1},\quad y\in\mathcal{H}(\Omega). \tag{122}$$

From this and by Hilbert - Schmidt theorem we receive the final assertion on solutions properties of the initial spectral problem (63) – (68).

Theorem 2. 1^0. *Spectral problem* (63) – (68) *has a discrete spectrum* $\{\lambda_k\}_{k=1}^{\infty}$ *consisting of finite - multiple eigenvalues* λ_k*, located on positive semiaxis* $\mathbb{R}_+$ *and having limit point* $\lambda = +\infty$.

2^0. *Eigenelements* $y_k = ((\Phi_{21})_k; \zeta_k)^t$, $k = 1, 2, \dots$, *form an orthogonal basis in Hilbert space* $\mathcal{H}(\Omega) = L_{2,\Omega_2} \oplus H_0$.

3^0. *Eigenvalues* λ_k *can be find as consecutive minima of functional* $F_1(\Phi_1; \Phi_2)$ *from* (70) *or as consecutive minima of the functional*

$$F_2(\Phi_1; \Phi_2) = \frac{c^2\rho_2 \int\limits_{\Omega_2} |\Delta\Phi_2|^2 \, d\Omega_2 + (\zeta, \zeta)_{B_\sigma}}{\rho_1 \int\limits_{\Omega_1} |\nabla\Phi_1|^2 \, d\Omega_1 + \rho_2 \int\limits_{\Omega_2} |\nabla\Phi_2|^2 \, d\Omega_2}, \tag{123}$$

see (53). *Both of these functionals must be considered on class of functions* $\Phi_1(x)$ *and* $\Phi_2(x)$ *for which the conditions*

$$\Delta\Phi_1 = 0 \ \ (in\ \Omega_1), \quad \frac{\partial\Phi_1}{\partial n} = 0 \ \ (on\ S_1), \quad \frac{\partial\Phi_2}{\partial n} = 0 \ \ (on\ S_2), \tag{124}$$

$$\zeta := \frac{\partial\Phi_1}{\partial n} = \frac{\partial\Phi_2}{\partial n} \ \ (on\ \Gamma), \quad \int\limits_{\Gamma} \zeta \, d\Gamma = 0, \quad \int\limits_{\Gamma} \Phi_1 \, d\Gamma = 0, \quad \int\limits_{\Omega_2} \Phi_2 \, d\Omega_2 = 0,$$

are fulfilled.

4^0. *For eigenfunctions of problem* (63) – (68) *the following equalities of orthogonalities are valid:*

$$(\mathcal{A}y_k, y_j)_{\mathcal{H}(\Omega)} = \sum_{m=1}^{2} \rho_m \int\limits_{\Omega_m} \nabla\Phi_{mk} \cdot \overline{\nabla\Phi_{mj}} \, d\Omega_m = \delta_{kj}, \tag{125}$$

$$(y_k, y_j)_{\mathcal{H}(\Omega)} = c^2\rho_2 \int\limits_{\Omega_2} \Delta\Phi_{2k} \cdot \overline{\Delta\Phi_{2j}} \, d\Omega_2 + \left(\frac{\partial\Phi_{1k}}{\partial n}\,|_{\Gamma}, \frac{\partial\Phi_{1j}}{\partial n}\,|_{\Gamma} \right)_{B_\sigma} = \lambda_k \delta_{kj}, \tag{126}$$

$$\left(B_\sigma^{-1} P_\Gamma(\rho_1\Phi_{1k} - \rho_2\Phi_{2k}), P_\Gamma(\rho_1\Phi_{1j} - \rho_2\Phi_{2j})\right)_0 + \rho_2 c^{-2} \int\limits_{\Omega_2} \Phi_{2k} \cdot \overline{\Phi_{2j}} \, d\Omega_2 = \lambda_k^{-1} \delta_{kj}. \tag{127}$$

Proof. 1^0. The first assertion is evident since (122) is a spectral problem for compact positive operator $\mathcal{A}$ and $\mu = \lambda^{-1}$.

2^0. The second one is also the corollary of Hilbert - Schmidt theorem.

3^0. Formulas (125) can be derived analogously to transforms (113) – (121). Then (126) follows from (122) and (125). Formulas (127) will be proved later (see Theorem 6). □

4. Variation principles for eigenvalues.

In the section, variation principles for eigenvalues of problem (63) – (68) are justified on the base of variation relations (70) and (123). The comparison of these principles are carried out in applications.

4.1. **The first variation principle.**

Consider once more spectral problem (63) – (68). For simplicity we put all physical constants (its are positive) to be equal to 1:

$$c = 1, \quad \rho_1 = 1, \quad \rho_2 = 1. \tag{128}$$

Then we will have spectral problem

$$\Delta\varphi_1 = 0 \ \text{ (in } \Omega_1), \quad -\Delta\varphi_2 = \lambda\varphi_2 \ \text{ (in } \Omega_2), \tag{129}$$

$$\zeta := \frac{\partial\varphi_1}{\partial n} = \frac{\partial\varphi_2}{\partial n} \ \text{ (on } \Gamma), \quad \frac{\partial\varphi_i}{\partial n} = 0 \ \text{ (on } S_i, i = 1, 2), \tag{130}$$

$$B_\sigma\zeta = \lambda P_\Gamma(\varphi_1 - \varphi_2) \ \text{ (on } \Gamma), \quad \int\limits_\Gamma \zeta \, d\Gamma = 0, \quad \lambda \int\limits_{\Omega_2} \varphi_2 \, d\Omega_2. \tag{131}$$

For this problem we receive (instead of (101)) the system of equations

$$A\eta = \lambda(\eta + T_2\zeta), \quad B_\sigma\zeta = \lambda(-P_\Gamma\gamma_2\eta + C\zeta), \tag{132}$$

$$\eta \in \mathcal{D}(A), \quad \zeta \in \mathcal{D}(B_\sigma), \tag{133}$$

for the same denotations of operators. By analogy with Theorem 2 we prove that problem (132), (133) has a discrete spectrum $\{\lambda_k\}_{k=1}^{\infty}$ consisting of finite multiple positive eigenvalues λ_k with limit point $\lambda = +\infty$.

Theorem 3. *The eigenvalues λ_k to problem* (129) – (131) *are consecutive minima of the variation relation*

$$F_1^0(\varphi_1; \varphi_2) := \frac{\sum_{k=1}^{2} \int\limits_{\Omega_k} |\nabla\varphi_k|^2 \, d\Omega_k}{\int\limits_{\Omega_2} |\varphi_2|^2 \, d\Omega_2 + \int\limits_\Gamma \left| B_\sigma^{-1/2} P_\Gamma(\varphi_1 - \varphi_2) \right| d\Gamma}. \tag{134}$$

This relation must be considered on functions $\varphi_k \in H^1_{\Omega_k}$ with the properties

$$\Delta\varphi_1 = 0 \ \ (in\ \Omega_1), \quad \frac{\partial\varphi_1}{\partial n} = 0 \ \ (on\ S_1), \quad \frac{\partial\varphi_2}{\partial n} = 0 \ \ (on\ S_2), \tag{135}$$

$$\frac{\partial\varphi_1}{\partial n} = \frac{\partial\varphi_2}{\partial n} =: \zeta \ \ (on\ \Gamma), \quad \int\limits_\Gamma \varphi_1 \, d\Gamma = 0, \quad \int\limits_{\Omega_2} \varphi_2 \, d\Omega_2 = 0. \tag{136}$$

Proof. 1^0. We use the substitution

$$A^{1/2}\eta =: \widetilde{\eta} \in \mathcal{D}(A^{1/2}) \tag{137}$$

in problem (132), (133). Then we have

$$A^{1/2}\widetilde{\eta} = \lambda(A^{-1/2}\widetilde{\eta} + T_2\zeta), \quad B_\sigma\zeta = \lambda(-P_\Gamma\gamma_2 A^{-1/2}\widetilde{\eta} + C\zeta). \tag{138}$$

If $\widetilde{\eta} \in L_{2,\Omega_2}$, then

$$A^{-1/2}\widetilde{\eta} + T_2\zeta \in \mathcal{D}(A^{1/2}) \tag{139}$$

because, by Lemma 4, the operator $A^{1/2}T_2$ is compact. Therefore we can apply the operator $A^{1/2}$ to the both parts of equation (138). It gives us the system of equations

$$\begin{pmatrix} A & 0 \\ 0 & B_\sigma \end{pmatrix} \begin{pmatrix} \widetilde{\eta} \\ \zeta \end{pmatrix} = \lambda \begin{pmatrix} I & Q \\ Q^* & C \end{pmatrix} \begin{pmatrix} \widetilde{\eta} \\ \zeta \end{pmatrix}, \tag{140}$$

$$Q^* = -P_\Gamma\gamma_2 A^{-1/2}, \quad Q = A^{1/2}T_2, \quad C = P_\Gamma(C_1 + C_2)P_\Gamma,$$

or

$$\mathcal{A}y = \lambda\mathcal{J}y, \quad y \in \mathcal{D}(A) \oplus \mathcal{D}(B_\sigma), \tag{141}$$

$$\begin{matrix} \mathcal{A} := \operatorname{diag}(A; B_\sigma) \gg 0, \\ y := (\widetilde{\eta}; \zeta)^t, \end{matrix} \qquad \mathcal{J} := \begin{pmatrix} I & Q \\ Q^* & C \end{pmatrix} > 0. \tag{142}$$

2^0. Problem (141) is equivalent to problem

$$\mathcal{J}^{-1}z = \lambda\mathcal{A}^{-1}z, \quad z = \mathcal{J}y, \tag{143}$$

and (143), in turn, is equivalent to problem

$$w = \lambda\mathcal{J}^{1/2}\mathcal{A}^{-1}\mathcal{J}^{1/2}w, \quad w = \mathcal{J}^{-1/2}z. \tag{144}$$

Since this problem, as problem (132), (133), has a discrete positive spectrum then bounded and selfadjoint operator $\mathcal{J}^{1/2}\mathcal{A}^{-1}\mathcal{J}^{1/2}$ is a compact positive operator. Therefore eigenvalues λ_k of this problem are consecutive minima of variation relation

$$\frac{(w,w)}{(\mathcal{A}^{-1/2}\mathcal{J}^{1/2}w, \mathcal{A}^{-1/2}\mathcal{J}^{1/2}w)} = \frac{(\mathcal{J}^{-1/2}z, z)}{(\mathcal{A}^{-1}z, z)} = \frac{(y,z)}{(\mathcal{A}^{-1}z, z)}. \tag{145}$$

3^0. We calculate numerator and denominator in (145) coming back to initial variables φ_1 and φ_2. We have

$$(y,z) = \begin{pmatrix} \widetilde{\eta} \\ \zeta \end{pmatrix} \cdot \begin{pmatrix} \widetilde{\eta} + Q\zeta \\ Q^*\eta + C\zeta \end{pmatrix} = \|\widetilde{\eta}\|^2_{\Omega_2} + (\widetilde{\eta}, Q\zeta)_{\Omega_2} + (\zeta, Q^*\widetilde{\eta})_0 + (\zeta, C\zeta)_0 =$$

$$= \left\|A^{1/2}\eta\right\|^2_{\Omega_2} + 2\operatorname{Re}(A^{1/2}\eta, A^{1/2}T_2\zeta)_{\Omega_2} + (P_\Gamma(\gamma_1T_1 - \gamma_2T_2)\zeta, \zeta)_0. \tag{146}$$

Since

$$\left\|A^{1/2}\eta\right\|^2_{\Omega_2} = \int_{\Omega_2} |\nabla\eta|^2\, d\Omega_2,\ \mathcal{D}(A^{1/2}) = H^1_{\Omega_2},\ \eta = \varphi_{21},\ T_2\zeta = \varphi_{22},\ T_1\zeta = \varphi_1, \tag{147}$$

then the right hand side in (146) is equal to

$$\int\limits_{\Omega_2} |\nabla\varphi_{21}|^2\, d\Omega_2 + 2\mathrm{Re}\int\limits_{\Omega_2} \nabla\varphi_{21}\cdot\overline{\nabla\varphi_{22}}\, d\Omega_2 + \int\limits_{\Omega_2} |\nabla\varphi_{22}|^2\, d\Omega_2 + \int\limits_{\Omega_1} |\nabla\varphi_1|^2\, d\Omega_1 =$$

$$= \int\limits_{\Omega_1} |\nabla\varphi_1|^2\, d\Omega_1 + \int\limits_{\Omega_2} |\nabla(\varphi_{21}+\varphi_{22})|^2\, d\Omega_2 = \sum_{k=1}^{2}\int\limits_{\Omega_k} |\nabla\varphi_k|^2\, d\Omega_k. \tag{148}$$

Calculating the denominator in (145) we have

$$(\mathcal{A}^{-1}z, z) = \begin{pmatrix} A^{-1} & 0 \\ 0 & B_\sigma^{-1} \end{pmatrix} \begin{pmatrix} \widetilde{\eta} + Q\zeta \\ Q^*\widetilde{\eta} + C\zeta \end{pmatrix} \cdot \begin{pmatrix} \widetilde{\eta} + Q\zeta \\ Q^*\widetilde{\eta} + C\zeta \end{pmatrix} =$$

$$= \begin{pmatrix} A^{-1}\left(\widetilde{\eta} + Q\zeta\right) \\ B_\sigma^{-1}\left(Q^*\widetilde{\eta} + C\zeta\right) \end{pmatrix} \cdot \begin{pmatrix} \widetilde{\eta} + Q\zeta \\ Q^*\widetilde{\eta} + C\zeta \end{pmatrix} = \left\| A^{-1/2}(\widetilde{\eta} + Q\zeta)\right\|^2_{\Omega_2} +$$

$$+ \left\| B_\sigma^{-1/2}(Q^*\widetilde{\eta} + C\zeta)\right\|_0^2 = \left\| B_\sigma^{-1/2}(-P_\Gamma\gamma_2\eta + P_\Gamma\left(\gamma_1 \mathrm{T}_1 - \gamma_2 T_2)\zeta\right)\right\|_0^2 +$$

$$+ \|\eta + T_2\zeta\|^2_{\Omega_2} = \|\varphi_{21}+\varphi_{22}\|^2_{\Omega_2} + \left\| B_\sigma^{-1/2} P_\Gamma(\gamma_1\varphi_1 - \gamma_2\varphi_{22} - \gamma_2\varphi_{21})\right\|_0^2 =$$

$$= \|\varphi_2\|^2_{\Omega_2} + \left\| B_\sigma^{-1/2} P_\Gamma(\gamma_1\varphi_1 - \gamma_2\varphi_2)\right\|_0^2 =$$

$$= \int\limits_{\Omega_2} |\varphi_2|^2\, d\Omega_2 + \int\limits_{\Gamma} \left| B_\sigma^{-1/2} P_\Gamma(\gamma_1\varphi_1 - \gamma_2\varphi_2)\right|^2 d\Gamma. \tag{149}$$

Now the variation principle (134) follows from (145), (148) and (149). Relations (135) – (136) take place because the functions φ_1 and φ_2 must be solutions to auxiliary Problems 1 and 2 (see Subsection 3.1) for the element $\zeta \in H_-$ (see formulas (83) – (88).) □

4.2. **The second variation principle.** We return to spectral problem (129) – (131). Our goal is to prove the second variation principle for eigenvalues λ of this problem. We check preliminary that numbers λ are coinside with values of the functional

$$F_2^0(\varphi_1;\varphi_2) := \frac{\displaystyle\int\limits_{\Omega_2} |\Delta\varphi_2|^2\, d\Omega_2 + \|\zeta\|^2_{B_\sigma}}{\displaystyle\sum_{k=1}^{2}\int\limits_{\Omega_k} |\nabla\varphi_k|^2\, d\Omega_k} \tag{150}$$

on solutions φ_1, φ_2 to problem (129) – (131) with taking into account relations (135), (136). Here quadratic functional $\|\zeta\|^2_{B_\sigma}$ is defined by (53).

To this end, we use the following relations that are valid for solutions to problem (129) – (131):

$$0 = -\int\limits_{\Omega_1} \Delta\varphi_1\cdot\varphi_1 d\Omega_1 = \int\limits_{\Omega_1} |\nabla\varphi_1|^2\, d\Omega_1 - \int\limits_{\Gamma} \frac{\partial\varphi_1}{\partial n}\cdot\varphi_1 d\Gamma = \int\limits_{\Omega_1} |\nabla\varphi_1|^2\, d\Omega_1 - \int\limits_{\Gamma} \zeta\varphi_1 d\Gamma;$$

$$-\int_{\Omega_2} \Delta\varphi_2 \cdot \Delta\varphi_2 \, d\Omega_2 = \lambda \int_{\Omega_2} \varphi_2 \Delta\varphi_2 \, d\Omega_2 = \lambda \left[-\int_{\Omega_2} |\nabla\varphi_2|^2 \; d\Omega_2 - \int_{\Gamma} \varphi_2 \frac{\partial\varphi_2}{\partial n} \, d\Gamma \right] =$$

$$= \lambda \left[-\int_{\Omega_2} |\nabla\varphi_2|^2 \; d\Omega_2 - \int_{\Gamma} \varphi_2 \zeta \, d\Gamma \right].$$

If we multiply he first relation by λ and subtract the second one we have

$$\lambda \sum_{k=1}^{2} \int_{\Omega_k} |\nabla\varphi_k|^2 \, d\Omega_k = \int_{\Omega_2} |\Delta\varphi_2|^2 \, d\Omega_2 + \lambda \int_{\Gamma} (\varphi_1 - \varphi_2)\zeta d\Gamma =$$

$$= \int_{\Omega_2} |\Delta\varphi_2|^2 \, d\Omega_2 + \lambda \int_{\Gamma} P_\Gamma(\varphi_1 - \varphi_2)\zeta d\Gamma = \int_{\Omega_2} |\Delta\varphi_2|^2 \; d\Omega_2 + \|\zeta\|^2_{B_\sigma}. \tag{151}$$

From this the variation relation (150) follows.

Theorem 4. *Eigenvalues λ to problem* (129) – (131) *are consecutive minima of variation relation* (150) *considered on functions $\varphi_k \in H^1_{\Omega_k}$ such that conditions* (135), (136) *are valid and, additionally, conditions*

$$\Delta\varphi_2 \in L_2(\Omega_2), \quad \zeta = \frac{\partial\varphi_1}{\partial n} \,|_\Gamma = \frac{\partial\varphi_2}{\partial n} \,|_\Gamma \in H = H_0 = L_{2,\Gamma}, \tag{152}$$

are valid also.

Proof. With account of (128) problem (107) has the form

$$y = \lambda \mathcal{A}_0 y, \quad y = (\psi; \varphi)^t \in \mathcal{H}(\Omega), \tag{153}$$

$$\mathcal{A}_0 = \begin{pmatrix} A^{-1} & A^{-1/2} Q B_\sigma^{-1/2} \\ B_\sigma^{-1/2} Q^* A^{-1/2} & B_\sigma^{-1/2} C B_\sigma^{-1/2} \end{pmatrix}, \qquad \begin{matrix} \psi = A\eta = A\varphi_{21}, \\ \varphi = B_\sigma^{1/2}\zeta. \end{matrix} \tag{154}$$

Here, as in problem (107), the matrix operator $\mathcal{A}_0$ is compact and positive (see Lemma 5 and Theorem 1). Therefore eigenvalues λ are consecutive minima of the variation relation

$$\frac{(y, y)}{(\mathcal{A}_0 y, y)} = \frac{\|\psi\|^2_{\Omega_2} + \|\varphi\|^2_0}{(\mathcal{A}_0 y, y)}. \tag{155}$$

Since, by definition (see Problem 3), $A\varphi_{21} = -\Delta\varphi_{21}$, $\varphi_{21} \in \mathcal{D}(A)$, then the numerator in (155) is equal to

$$\int_{\Omega_2} |\Delta\varphi_{21}|^2 \; d\Omega + \int_{\Gamma} \left| B_\sigma^{1/2} \zeta \right|^2 \; d\Gamma = \int_{\Omega_2} |\Delta\varphi_2|^2 \; d\Omega_2 + \|\zeta\|^2_{B_\sigma}, \tag{156}$$

because $\varphi_{21} = \varphi_2 - \varphi_{22}$, $\Delta\varphi_{22} = 0$ (see Problem 2).

As for the denominator in (155) then the quadratic form $(\mathcal{A}_0 y, y)$ can be derived by the same way as it was done in proof of Theorem 1. Taking into account (128) and using formula (121), we have

$$(\mathcal{A}_0 y, y) = \sum_{k=1}^{2} \int_{\Omega_k} |\nabla\varphi_k|^2 \, d\Omega_k, \tag{157}$$

and from (155) – (157) the variation principle (150) follows. □

Remark 1. Conditions (152) in Theorem 4 (i.e., in the second variation principle, see (150)) are sufficiently restrictive and its are connected with smoothness of functions $\varphi_i(x)$ in domains Ω_i with nonsmooth boundaries $\partial\Omega_i$, $i = 1, 2$. In the first variation principle (see Theorem 3) these conditions are absent. □

4.3. **Comparison of the variation principles.** Note at first that numbers $\mu_k := \lambda_k^{-1}$ in problem (129) – (131) are consecutive maxima of the variation relation (see (134))

$$F_3^0(\varphi_1; \varphi_2) := \frac{\int_{\Omega_2} |\varphi_2|^2 \, d\Omega_2 + \int_{\Gamma} \left| B_\sigma^{-1/2} P_\Gamma(\varphi_1 - \varphi_2) \right|^2 d\Gamma}{\sum_{k=1}^{2} \int_{\Omega_k} |\nabla\varphi_k|^2 \, d\Omega_k}. \tag{158}$$

This fact follows as from Theorem 3 as from equation (122).

Now we will carry out the comparison of the variation principles on the base of functionals $F_1^0(\varphi_1; \varphi_2)$ from (134), $F_2^0(\varphi_1; \varphi_2)$ from (150) and $F_3^0(\varphi_1; \varphi_2)$ from (158) if we will use the Ritz method of numerical calculations of eigenvalues and eigenfunctions for problem (129) – (131).

As it follows from Theorem 3, one can find the eigenvalues λ to problem (129) – (131) considering the variation problem on minimum for the functional

$$I(\varphi_1; \varphi_2) := \sum_{k=1}^{2} \int_{\Omega_k} |\nabla\varphi_k|^2 \, d\Omega_k \tag{159}$$

under the additional condition

$$K(\varphi_1; \varphi_2) := \int_{\Omega_2} |\varphi_2|^2 \, d\Omega_2 + \int_{\Gamma} \left| B_\sigma^{-1/2} P_\Gamma(\varphi_1 - \varphi_2) \right|^2 d\Gamma = \text{const} > 0. \tag{160}$$

Instead of (159), (160) one can consider the problem on unconditional extremum for the functional

$$I_*(\varphi_1; \varphi_2) := I(\varphi_1; \varphi_2) - \lambda K(\varphi_1; \varphi_2) \tag{161}$$

with taking into account connections (135), (136).

It follows from Theorem 4 that one can find the eigenvalues λ solving the problem on minimum for the functional

$$M(\varphi_1;\varphi_2) := \int\limits_{\Omega_2} |\Delta\varphi_2|^2 \, d\Omega_2 + \|\zeta\|^2_{B_\sigma} \tag{162}$$

under the additional condition

$$I(\varphi_1;\varphi_2) = \text{const} > 0, \tag{163}$$

i.e., the problem on unconditional extremum for the functional

$$M_*(\varphi_1;\varphi_2) := M(\varphi_1;\varphi_2) - \lambda I(\varphi_1;\varphi_2). \tag{164}$$

Here one must carry out variations in the class of functions such that conditions (135), (136), (152) must be valid.

Both of these approaches for functionals (161) and (164) on the base of Ritz method have the following restrictive fact: coordinate (basis) functions that approximate the solution $\varphi_1(x)$ must be harmonic functions in the region Ω_1 and Newmann condition must be valid on the surface S_1 for them. We can not take into account this restriction if we will use the variation principle on the base of functional $F_3^0(\varphi_1;\varphi_2)$ from (158).

Theorem 5. *In problem* (129) - (131) *one can find numbers* $\mu = \lambda^{-1}$ *by Ritz method considering the problem on maximum of the functional* $K(\varphi_1;\varphi_2)$ *under additional condition* $I(\varphi_1;\varphi_2) = const > 0$ *or in the problem on unconditional extremum for the functional*

$$K_*(\varphi_1;\varphi_2) := K(\varphi_1;\varphi_2) - \mu I(\varphi_1;\varphi_2). \tag{165}$$

In this, it is sufficient to carry out the variation in (164) *in class of functions* $\varphi_i \in H^1_{\Omega_i}$, $i = 1, 2$.

Here conditions (135), (136) *for functional are natural, i.e., its are valid automatically for solutions to problem* (129) – (131) *with* $\lambda = \mu^{-1}$.

Proof. Let $\delta\varphi_i(x)$ be arbitrary functions from $H^1_{\Omega_i}$, $i = 1, 2$. Then

$$\int\limits_{\Gamma} \delta\varphi_1 \, d\Gamma = 0, \quad \int\limits_{\Omega_2} \delta\varphi_2 \, d\Omega_2 = 0. \tag{166}$$

Calculating variation of the functional K_* on these functions and equating it to zero we have

$$\frac{1}{2}\,\delta K_*(\varphi_1,\varphi_2;\delta\varphi_1,\delta\varphi_2) = \int\limits_{\Omega_2} \varphi_2\delta\varphi_2 \, d\Omega_2 + \int\limits_{\Gamma} B_\sigma^{-1} P_\Gamma(\varphi_1-\varphi_2) P_\Gamma(\delta\varphi_1 - \delta\varphi_2) \, d\Gamma -$$

$$-\mu \sum_{k=1}^{2} \int\limits_{\Omega_k} \nabla\varphi_k \cdot \nabla\delta\varphi_k \, d\Omega_k = \int\limits_{\Omega_2} \varphi_2\delta\varphi_2 \, d\Omega_2 + \int\limits_{\Gamma} B_\sigma^{-1} P_\Gamma(\varphi_1-\varphi_2)\delta\varphi_1 \, d\Gamma -$$

$$-\int\limits_{\Gamma} B_\sigma^{-1} P_\Gamma(\varphi_1-\varphi_2)\delta\varphi_2\,d\Gamma - \mu\left(-\int\limits_{\Omega_1}\Delta\varphi_1\delta\varphi_1\,d\Omega_2 + \int\limits_{S_1}\frac{\partial\varphi_1}{\partial n}\delta\varphi_1\,dS_1\right) -$$

$$-\mu\left(\int\limits_{\Gamma}\frac{\partial\varphi_1}{\partial n}\delta\varphi_1\,d\Gamma - \int\limits_{\Omega_2}\Delta\varphi_2\delta\varphi_2\,d\Omega_2 + \int\limits_{S_2}\frac{\partial\varphi_2}{\partial n}\delta\varphi_2\,dS_2 - \int\limits_{\Gamma}\frac{\partial\varphi_2}{\partial n}\delta\varphi_2\,d\Gamma\right) =$$

$$= \int\limits_{\Omega_2}(\varphi_2+\mu\Delta\varphi_2)\delta\varphi_2\,d\Omega_2 + \mu\int\limits_{\Omega_1}\Delta\varphi_1\delta\varphi_1\,d\Omega_1 - \mu\int\limits_{S_1}\frac{\partial\varphi_1}{\partial n}\delta\varphi_1\,dS_1 -$$

$$-\mu\int\limits_{S_2}\frac{\partial\varphi_2}{\partial n}\delta\varphi_2\,dS_2 + \int\limits_{\Gamma}\left(B_\sigma^{-1}P_\Gamma(\varphi_1-\varphi_2) - \mu\frac{\partial\varphi_1}{\partial n}\right)\delta\varphi_1\,d\Gamma -$$

$$-\int\limits_{\Gamma}\left(B_\sigma^{-1}P_\Gamma(\varphi_1-\varphi_2) - \mu\frac{\partial\varphi_2}{\partial n}\right)\delta\varphi_2\,d\Gamma = 0. \tag{167}$$

From this one can prove sequentially the following facts.

1^0. If $\delta\varphi_2 \equiv 0$ in Ω_2 and $\delta\varphi_1$ is a compactly supported (finitary) function in Ω_1 then (with account of density property of finitary functions in $L_2(\Omega_1)$) we have that the equation $\Delta\varphi_1 = 0$ (in Ω_1) is valid for φ_1.

2^0. Putting on $\delta\varphi_2 \equiv 0$, $\delta\varphi_1 \equiv 0$ (on Γ) and using the fact that $\delta\varphi_1$ is arbitrary on S_1, we have the boundary condition $\dfrac{\partial\varphi_1}{\partial n} = 0$ (on S_1).

3^0. Putting on $\delta\varphi_2 \equiv 0$ and using the fact, that $\delta\varphi_1$ is an arbitrary function on Γ with $\int\limits_{\Gamma}\delta\varphi_1\,d\Gamma = 0$, we receive the condition

$$B_\sigma^{-1}P_\Gamma(\varphi_1-\varphi_2) - \mu\frac{\partial\varphi_1}{\partial n} = 0 \text{ (on } \Gamma).$$

(More concrete, here the right hand side is equal to constant and it is equal to zero because

$$\int\limits_{\Gamma} B_\sigma^{-1}P_\Gamma(\varphi_1-\varphi_2)\,d\Gamma = 0, \qquad \int\limits_{\Gamma}\frac{\partial\varphi_1}{\partial n}\,d\Gamma = 0.)$$

4^0. Let now $\delta\varphi_2$ be finitary. Then from (167) (with account of received relations) we calculate that

$$\varphi_2 + \mu\Delta\varphi_2 = 0 \text{ (in } \Omega_2).$$

5^0. If $\delta\varphi_2 \equiv 0$ (on Γ) then we have $\dfrac{\partial\varphi_2}{\partial n} = 0$ (on S_2).

6^0. At last, if $\delta\varphi_2$ is an arbitrary function on Γ then we have the condition

$$\mu\frac{\partial\varphi_2}{\partial n} - B_\sigma^{-1}P_\Gamma(\varphi_1-\varphi_2) = 0 \text{ (on } \Gamma),$$

(Indeed, since $\int\limits_{\Omega_2} \delta\varphi_2 \, d\Omega_2 = 0$, then for the first time we have

$$\mu \frac{\partial \varphi_2}{\partial n} - B_\sigma^{-1} P_\Gamma (\varphi_1 - \varphi_2) = \text{const.}$$

But

$$-\int\limits_{\Omega_2} \Delta\varphi_2 \, d\Omega_2 = \lambda \int\limits_{\Omega_2} \varphi_2 \, d\Omega_2 = \ldots = \int\limits_{\Gamma} \frac{\partial \varphi_2}{\partial n} \, d\Gamma = 0,$$

and therefore above constant is equal to zero.)

Thus, solutions φ_1 and φ_2, corresponding to stationar values of functional (164) for $\mu = \lambda^{-1}$, are solutions to spectral problem (129) – (131). □

5. On orthogonal basis property of the eigenfunctions.

In this section, properties of orthogonal basis for the system of eigenfunctions to problem (63) – (68) or (129) – (131) are studied. We define more exactly Hilbert spaces where these eigenfunctions form an orthogonal basis.

5.1. **Some additional assertions.** In the space $H^1_{\Omega_1}$ (see Subsection 3.1) we introduce the subspace $H^1_{h,S_1}(\Omega_1)$ of harmonic functions that are formed by weak solutions to the auxiliary Problem 1 for all $\zeta \in \left(H_\Gamma^{1/2}\right)^*$:

$$H^1_{h,S_1}(\Omega_1) := \left\{ \varphi \in H^1_{\Omega_1} : \ \varphi = T_1\zeta, \ \forall\, \zeta \in \left(H_\Gamma^{1/2}\right)^* \right\}. \tag{168}$$

It follows from ([9], p. 106) that subspace

$$H^1_{0,\Gamma}(\Omega_1) := \left\{ \psi \in H^1_{\Omega_1} : \ \psi \equiv 0 \ \text{ on } \Gamma \right\} \tag{169}$$

is an orthogonal complement to $H^1_{h,S_1}(\Omega_1)$ in the space $H^1_{\Omega_1}$.

Introduce also the space

$$\begin{aligned} \mathcal{H}^1(\Omega) := \Big\{ \varphi = (\varphi_1; \varphi_2) : \ \varphi_2 \in H^1_{\Omega_2}, \quad \varphi_1 \in H^1_{h,S_1}(\Omega_1), \\ \frac{\partial \varphi_2}{\partial n} \, |_\Gamma = \frac{\partial \varphi_1}{\partial n} \, |_\Gamma =: \zeta, \quad \frac{\partial \varphi_2}{\partial n} \, |_{S_2} = 0 \Big\} \end{aligned} \tag{170}$$

with the norm

$$\|\varphi\|^2_{1,\Omega} := \sum_{k=1}^{2} \int\limits_{\Omega_k} |\nabla \varphi_k|^2 \, d\Omega_k; \tag{171}$$

this space is connected naturally with problem (129) – (131).

Lemma 6. *Any element $\varphi = (\varphi_1; \varphi_2) \in \mathcal{H}^1(\Omega)$ has a representation*

$$\varphi_1 = T_1\zeta, \quad \varphi_2 = T_2\zeta + A^{-1}f, \quad \zeta \in \left(H_\Gamma^{1/2}\right)^*, \quad f \in \left(H^1_{\Omega_2}\right)^*, \tag{172}$$

where T_1, T_2 and A are operators of auxiliary Problems 1 – 3 (see Subsection 3.1). The operator

$$\mathcal{J} := \begin{pmatrix} A^{-1} & T_2 \\ 0 & T_1 \end{pmatrix} : \ \left(H^1_{\Omega_2}\right)^* \times \left(H^{1/2}_\Gamma\right)^* \longrightarrow \mathcal{H}^1(\Omega) \subset H^1_{\Omega_2} \times H^1_{h,S_1}(\Omega_1) \tag{173}$$

determines one-to-one correspondence between elements $(f;\zeta)^t$ and $(\varphi_2;\varphi_1)^t$, it is bounded and has bounded inverse.

Proof. 1^0. Let $f \in \left(H^1_{\Omega_2}\right)^*$, $\zeta \in \left(H^{1/2}_\Gamma\right)^*$. Then, according to solutions properties of auxiliary Problem 1, we have $\varphi_1 := T_1\zeta \in H^1_{h,S_1}(\Omega_1)$. By Problem 2, we have analogously $\varphi_{22} := T_2\zeta \in H^1_{h,S_2}(\Omega_2) \subset H^1_{\Omega_2}$. Introduce also, by Problem 3, an element $\varphi_{21} := A^{-1}f \in H^1_{\Omega_2}$. Then $\varphi_2 := \varphi_{21} + \varphi_{22} \in H^1_{\Omega_2}$ and therefore

$$\varphi := (\varphi_2;\varphi_1)^t \in H^1_{\Omega_2} \times H^1_{h,S_1}(\Omega_1), \quad \frac{\partial\varphi_2}{\partial n} = \frac{\partial\varphi_1}{\partial n} = \zeta \ (\text{on } \Gamma),$$

i.e., $\varphi \in \mathcal{H}^1(\Omega)$.

Hence, representations (172) and (173) are proved. Remark now that in (173) the operator T_1 acts boundedly from $\left(H^{1/2}_\Gamma\right)^*$ onto $H^1_{h,S_1}(\Omega_1)$, the operator T_2 acts boundedly from $\left(H^{1/2}_\Gamma\right)^*$ onto $H^1_{h,S_2}(\Omega_2) \subset H^1_{\Omega_2}$ and the operator A^{-1} acts boundedly from $\left(H^1_{\Omega_2}\right)^*$ onto $H^1_{\Omega_2}$. Therefore the operator matrix $\mathcal{J}$ from (173) is bounded from $\left(H^1_{\Omega_2}\right)^* \times \left(H^{1/2}_\Gamma\right)^*$ into $\mathcal{H}^1(\Omega)$.

2^0. Conversely, let

$$\varphi_2 \in H^1_{\Omega_2}, \quad \varphi_1 \in H^1_{h,S_1}(\Omega_1), \quad \frac{\partial\varphi_1}{\partial n} = \frac{\partial\varphi_2}{\partial n} \ (\text{on } \Gamma).$$

Then $\zeta := T_1^{-1}\varphi_1 = \dfrac{\partial\varphi_1}{\partial n}\ |_\Gamma \in \left(H^{1/2}_\Gamma\right)^*$ (see (86)). Introduce $\varphi_{22} := T_2\zeta = T_2T_1^{-1}\varphi_1 \in H^1_{h,S_2}(\Omega_2) \subset H^1_{\Omega_2}$. Then

$$\varphi_{21} := \varphi_2 - \varphi_{22} = \varphi_2 - T_2T_1^{-1}\varphi_1 \in H^1_{\Omega_2} = \mathcal{R}(A^{-1}) = \mathcal{D}(A),$$

and therefore

$$f := A(\varphi_2 - \varphi_{22}) = A\varphi_2 - AT_2T_1^{-1}\varphi_1 \in \left(H^1_{\Omega_2}\right)^*.$$

Finally, we have

$$\begin{pmatrix} f \\ \zeta \end{pmatrix} = \begin{pmatrix} A & -AT_2T_1^{-1} \\ 0 & T_1^{-1} \end{pmatrix} \begin{pmatrix} \varphi_2 \\ \varphi_1 \end{pmatrix} \in \left(H^1_{\Omega_2}\right)^* \times \left(H^{1/2}_\Gamma\right)^*, \tag{174}$$

where the operator

$$\mathcal{J}^{-1} = \begin{pmatrix} A & -AT_2T_1^{-1} \\ 0 & T_1^{-1} \end{pmatrix} : H^1_{\Omega_2} \times H^1_{h,S_1}(\Omega_1) \longrightarrow \left(H^1_{\Omega_2}\right)^* \times \left(H^{1/2}_\Gamma\right)^* \tag{175}$$

is bounded because here all entries are bounded operators. □

5.2. **Orthogonal basis properties for the system of eigenfunctions.** On the base of above proved facts, we will prove here orthogonal basis property for eigenfunctions of problem (129) – (131) and initial spectral problem (63) – (68).

Theorem 6. *Eigenfunctions*

$$\{\varphi_k\}_{k=1}^{\infty} := \{\varphi_{2k}; \varphi_{1k}\}_{k=1}^{\infty}$$

of problem (129) – (131) *form an orthogonal basis in the space* $\mathcal{H}^1(\Omega)$ *(see* (170)*). Respectively, eigenfunctions* $\Phi_k := (\Phi_{2k}; \Phi_{1k})$, $k = 1, 2, \ldots$, *of problem* (63) – (68) *form an orthogonal basis in the space* $\mathcal{H}_1(\Omega; \rho)$ *with the norm*

$$\|\Phi\|^2_{1,\Omega,\rho} := \sum_{m=1}^{2} \rho_m \int\limits_{\Omega_m} |\nabla \Phi_m|^2 \, d\Omega_m. \tag{176}$$

In this, for eigenfunctions $\{\varphi_k\}_{k=1}^{\infty}$ *of problem* (129) – (131) *the following formulas*

$$\left.\begin{aligned} \sum_{m=1}^{2} \int\limits_{\Omega_m} \nabla\varphi_{mk} \cdot \nabla\varphi_{mj} \, d\Omega_m = \delta_{kj}, \\ \int\limits_{\Omega_2} \Delta\varphi_{2k} \cdot \Delta\varphi_{2j} \, d\Omega_2 + \left(\frac{\partial\varphi_{1k}}{\partial n}\,|_\Gamma, \frac{\partial\varphi_{1j}}{\partial n}\,|_\Gamma \right)_{B_\sigma} = \lambda_k \delta_{kj}, \\ \int\limits_{\Omega_2} \varphi_{2k} \cdot \varphi_{2j} \, d\Omega_2 + \int\limits_{\Gamma} \left(B_\sigma^{-1} P_\Gamma(\varphi_{1k} - \varphi_{2k}) \right) (\varphi_{1j} - \varphi_{2j}) \, d\Gamma = \lambda_k^{-1} \delta_{kj}, \end{aligned}\right\} \tag{177}$$

are valid, and for eigenelements $\{\Phi_k\}_{k=1}^{\infty}$ *of problem* (63) – (68) *formulas orthogonality* (125) – (127) *hold.*

Proof. It is evident that we can prove only the first assertion of the theorem, i.e., properties for functions $\{\varphi_k\}_{k=1}^{\infty}$. Proof of corresponding properties for functions $\{\Phi_k\}_{k=1}^{\infty}$ is the same.

As it follows from proof of Theorem 4, eigenelements $y_k = \left(-\Delta\varphi_{2k}; B_\sigma^{1/2} \left(\frac{\partial\varphi_{1k}}{\partial n} \right)_\Gamma \right)^t$ of problem (153) – (154) form an orthogonal basis in the space $\mathcal{H}(\Omega) = L_{2,\Omega_2} \oplus H_0$. By (153) and (157),

$$(\mathcal{A}_0 y_k, y_l) = \sum_{m=1}^{2} \rho_m \int\limits_{\Omega_m} \nabla\varphi_{mk} \cdot \nabla\varphi_{ml} \, d\Omega_m = 0 \ \ (k \neq l), \tag{178}$$

and if $(\mathcal{A}_0 y_k, y_l) = \delta_{kl}$, then the system of eigenelements $\{(\varphi_{2k}; \varphi_{1k})\}_{k=1}^{\infty}$ to problem (129) – (131) is orthonormal in the space $\mathcal{H}^1(\Omega)$. We will prove now that this system form an orthogonal basis in $\mathcal{H}^1(\Omega)$.

Since the operators $\mathcal{J}$ and $\mathcal{J}^{-1}$, by Lemma 6, are bounded, it is sufficient to check that the set of elements

$$y_k = \left(-\Delta\varphi_{2k};\ B_\sigma^{1/2}\left(\frac{\partial\varphi_{1k}}{\partial n}\right)_\Gamma\right)^t,\ k = 1, 2, \ldots, \tag{179}$$

is complete in the space $\left(H^1_{\Omega_2}\right)^* \times \left(H^{1/2}_\Gamma\right)^*$. Indeed, in the case the system of elements $\{(\varphi_{2k}; \varphi_{1k})\}_{k=1}^\infty$ will be complete in $\mathcal{H}^1(\Omega)$ and orthogonal, i.e., it will be an orthogonal basis in $\mathcal{H}^1(\Omega)$.

Let $\varphi^0 = (\varphi_2^0; \varphi_1^0)^t$ be an arbitrary element from $\mathcal{H}^1(\Omega)$. Then, by Lemma 6, the element

$$\psi^0 := \left(-\Delta\varphi_2^0; \left(\frac{\partial\varphi_1^0}{\partial n}\right)_\Gamma\right)^t = \mathcal{J}^{-1}\varphi^0 \in \left(H^1_{\Omega_2}\right)^* \times \left(H^{1/2}_\Gamma\right)^*. \tag{180}$$

Since the space L_{2,Ω_2} and $L_{2,\Gamma} = H_0$ have equipments, i.e.,

$$H^1_{\Omega_2} \subset L_{2,\Omega_2} \subset \left(H^1_{\Omega_2}\right)^*, \quad H^{1/2}_\Gamma \subset L_{2,\Gamma} \subset \left(H^{1/2}_\Gamma\right)^*, \tag{181}$$

then

$$\left(H^1_{\Omega_2}\right)^* \times \left(H^{1/2}_\Gamma\right)^* \supset L_{2,\Omega_2} \oplus H_0 = \mathcal{H}(\Omega) \tag{182}$$

and $\mathcal{H}(\Omega)$ is dense in $\left(H^1_{\Omega_2}\right)^* \times \left(H^{1/2}_\Gamma\right)^*$. Therefore for any $\varepsilon > 0$ there exists an element $\widetilde{\psi}^0 \in \mathcal{H}(\Omega)$ such that

$$\left\|\psi^0 - \widetilde{\psi}^0\right\|_{(H^1_{\Omega_2})^* \times \left(H^{1/2}_\Gamma\right)^*} < \varepsilon/2. \tag{183}$$

Further, for any element $u \in \mathcal{H}(\Omega)$ the inequality

$$\|u\|_{\left(H^1_{\Omega_2}\right)^* \times \left(H^{1/2}_\Gamma\right)^*} \leqslant c\,\|u\|_{\mathcal{H}(\Omega)} \tag{184}$$

holds since the embedding operator from $\mathcal{H}(\Omega)$ into $\left(H^1_{\Omega_2}\right)^* \times \left(H^{1/2}_\Gamma\right)^*$ is bounded (and even compact). Since elements $\{y_k\}_{k=1}^\infty$ from (179) form an orthogonal basis in $\mathcal{H}(\Omega)$ and therefore form a complete system, then one can take a number $N = N(\varepsilon) \in \mathbb{N}$ and coefficients c_k, $k = 1, \ldots, N(\varepsilon)$, such that

$$\left\|\widetilde{\psi}^0 - \sum_{k=1}^{N(\varepsilon)} c_k y_k\right\|_{\mathcal{H}(\Omega)} < \frac{\varepsilon}{2c}, \tag{185}$$

where $c > 0$ is a constant from (184). Then, by (184) and (185), we have

$$\left\|\psi^0 - \sum_{k=1}^{N(\varepsilon)} c_k y_k\right\|_{(H^1_{\Omega_2})^* \times (H^{1/2}_\Gamma)^*} = \left\|(\psi^0 - \widetilde{\psi}^0) + \left(\widetilde{\psi}^0 - \sum_{k=1}^{N(\varepsilon)} c_k y_k\right)\right\|_{\left(H^1_{\Omega_2}\right)^* \times \left(H^{1/2}_\Gamma\right)^*} <$$

$$< \frac{\varepsilon}{2} + \left\|\widetilde{\psi}^0 - \sum_{k=1}^{N(\varepsilon)} c_k y_k\right\|_{\left(H^1_{\Omega_2}\right)^* \times \left(H^{1/2}_\Gamma\right)^*} < \frac{\varepsilon}{2} + c\left\|\widetilde{\psi}^0 - \sum_{k=1}^{N(\varepsilon)} c_k y_k\right\|_{\mathcal{H}(\Omega)} < \varepsilon,$$

i.e., the system of elements from (179) is complete in $\left(H^1_{\Omega_2}\right)^* \times \left(H^{1/2}_\Gamma\right)^*$. □

On the base of above proved assertions, we will prove corresponding basis properties of and completeness for the system of eigenfunctions to spectral problem generated by initial boundary value problem (8) – (15).

We will consider solutions to homogeneous problem (8) – (14) in the form

$$\vec{w}_i(t,x) = \vec{w}_i(x)e^{i\omega t}, \quad p_i(t,x) = p_i(x)e^{i\omega t}, \quad i = 1,2, \tag{186}$$

where ω is a frequency of oscillations and $\vec{w}_i(x)$, $p_i(x)$ are so called amplitude functions (modes of oscillations). We have the following spectral problem for these functions:

$$\lambda\vec{w}_1 = \frac{1}{\rho_1}\nabla p_1, \quad \operatorname{div}\vec{w}_1 = 0 \text{ (in } \Omega_1), \quad \vec{w}_1\cdot\vec{n} = 0 \text{ (on } S_1), \quad \lambda = \omega^2, \tag{187}$$

$$\lambda\vec{w}_2 = \frac{1}{\rho_2}\nabla p_2, \quad p_2 + \rho_2 c^2 \operatorname{div}\vec{w}_2 = 0 \text{ (in } \Omega_2), \quad \vec{w}_2\cdot\vec{n} = 0 \text{ (on } S_2), \tag{188}$$

$$\vec{w}_1\cdot\vec{n} = \vec{w}_2\cdot\vec{n} =: \zeta, \quad P_\Gamma(p_1 - p_2) = B_\sigma\zeta \text{ (on } \Gamma). \tag{189}$$

This problem is equivalent to problem (63) – (68) since

$$\vec{w}_i(x) = \nabla\Phi_i(x), \ i = 1,2.$$

On the base of orthogonal decompositions (23) and (34) introduce subspace

$$\vec{G}(\Omega) := \vec{G}(\Omega_2) \oplus \vec{G}_{h,S_1}(\Omega_1) := \Big\{\vec{w} := (\vec{w}_2;\vec{w}_1) : \ \vec{w}_2 = \nabla\Phi_2 \in \vec{G}(\Omega_2),$$

$$\vec{w}_1 = \nabla\Phi_1 \in \vec{G}_{h,S_1}(\Omega_1), \quad \vec{w}_1\cdot\vec{n} = \vec{w}_2\cdot\vec{n} =: \zeta \text{ (on } \Gamma), \quad \frac{\partial\Phi_2}{\partial n} = 0 \text{ (on } S_2)\Big\} \tag{190}$$

in the space $\vec{L}_2(\Omega_1) \oplus \vec{L}_2(\Omega_2)$ with scalar product

$$(\vec{w},\vec{v}) := \sum_{k=1}^{2} \rho_k \int_{\Omega_k} \vec{w}_k\cdot\vec{v}_k \, d\Omega_k. \tag{191}$$

It is evident that solutions $\vec{w} = (\vec{w}_2;\vec{w}_1)$ to problem (187) – (189) must belong to the space $\vec{G}(\Omega)$.

Theorem 7. *Eigenfunctions $\vec{w}_k = (\vec{w}_{2k};\vec{w}_{1k}) = (\nabla\Phi_{2k};\nabla\Phi_{1k})$, $k = 1,2,\ldots$, to problem* (187) – (189)*, corresponding to nonzero eigenvalues λ_k, form an orthogonal basis in the subspace $\vec{G}(\Omega)$.*

Proof. By Theorem 6, eigenfunctions $\{(\Phi_{2k};\Phi_{1k})\}_{k=1}^\infty$ of problem (63) – (68) form an orthogonal basis in the space $\mathcal{H}^1(\Omega;\rho)$ with squared norm (176). It follows from (190) and (191) that there exists isometric isomorphism between elements of spaces $\mathcal{H}^1(\Omega;\rho)$ and $\vec{G}(\Omega)$.

Indeed, any element $(\nabla\Phi_2; \nabla\Phi_1) \in \vec{G}(\Omega)$ is defined uniquely by the element $(\Phi_2; \Phi_1) \in \mathcal{H}^1(\Omega; \rho)$. Conversely, an element $(\Phi_2; \Phi_1)$ is uniquely defined by $(\nabla\Phi_2; \nabla\Phi_1) \in \vec{G}(\Omega)$ because we must take into account conditions (26) and (35):

$$\int\limits_{\Gamma} \Phi_1 \, d\Gamma = 0, \qquad \int\limits_{\Omega_2} \Phi_2 \, d\Omega_2 = 0. \tag{192}$$

Finally, for arbitrary $(\vec{w}_2; \vec{w}_1)$, $(\vec{v}_2; \vec{v}_1)$ from $\vec{G}(\Omega)$, $\vec{w}_i = \nabla\Phi_i$, $\vec{v}_i = \nabla\Psi_i$, $i = 1, 2$, we have

$$((\vec{w}_2; \vec{w}_1), (\vec{v}_2; \vec{v}_1))_{\vec{G}(\Omega)} = ((\nabla\Phi_2; \nabla\Phi_1), (\nabla\Psi_2; \nabla\Psi_1))_{\vec{G}(\Omega)} =$$

$$= \sum_{k=1}^{2} \rho_k \int\limits_{\Omega_k} \vec{w}_k \cdot \vec{v}_k \, d\Omega_k = \sum_{k=1}^{2} \rho_k \int\limits_{\Omega_k} \nabla\Phi_k \cdot \nabla\Psi_k \, d\Omega_k = ((\Phi_2; \Phi_1), (\Psi_2; \Psi_1))_{\mathcal{H}_1(\Omega,\rho)}. \tag{193}$$

It proves the theorem. $\square$

5.3. **Some limit cases.** Comming back to variation principles for eigenvalues $\lambda = \omega^2$ in problem (63) – (68) (see theorems 3 – 5) we remark once more that these eigenvalues can be find as consecutive minima of the functional

$$F_1(\Phi_1; \Phi_2) = \frac{\sum\limits_{k=1}^{2} \rho_k \int\limits_{\Omega_k} |\nabla\Phi_k|^2 \, d\Omega_k}{\rho_2 c^{-2} \int\limits_{\Omega_2} |\Phi_2|^2 \, d\Omega_2 + \left\| B_\sigma^{-1/2} P_\Gamma(\rho_1\Phi_1 - \rho_2\Phi_2) \right\|_0^2} \tag{194}$$

or the functional

$$F_2(\Phi_1; \Phi_2) = \frac{c^2 \rho_2 \int\limits_{\Omega_2} |\Delta\Phi_2|^2 \, d\Omega_2 + \left\| \left(\frac{\partial\Phi_1}{\partial n} \right)_\Gamma \right\|_{B_\sigma}^2}{\sum\limits_{k=1}^{2} \rho_k \int\limits_{\Omega_k} |\nabla\Phi_k|^2 \, d\Omega_k} \tag{195}$$

on corresponding classes of functions Φ_1 and Φ_2, see conditions (135), (136) for $\varphi_i = \Phi_i$, $i = 1, 2$.

Consider limit problems in variation relations (194), (195). These problems correspond to limit values of physical parameters in studied hydrodynamical system „fluid – gas".

1^0. If the density of a gas tends to zero, $\rho_2 \longrightarrow 0$, then in limit we have the well-known problem on small oscillations of a capillary ideal fluid in an open vessel (see, for instance, [9], p. 207). Then

$$F_1 = F_1(\Phi_1) = \frac{\rho_1 \int\limits_{\Omega_1} |\nabla\Phi_1|^2 \, d\Omega_1}{\left\| B_\sigma^{-1/2} \rho_1 \Phi_1 \right\|_0^2}, \tag{196}$$

$$F_2 = F_2(\Phi_1) = \frac{\left\| \left(\dfrac{\partial \Phi_1}{\partial n} \right)_\Gamma \right\|^2_{B_\sigma}}{\rho_1 \displaystyle\int_{\Omega_1} |\nabla \Phi_1|^2 \, d\Omega_1}. \tag{197}$$

2^0. If the velocity of a sound tends to infinity, $c^2 \longrightarrow \infty$, then in limit we have a problem on small oscillation of two capillary ideal fluids with densities ρ_1 and ρ_2 (see [9], p. 212). Then we can put $c^{-2} = 0$ into functional (194):

$$F_1(\Phi_1; \Phi_2) \mid_{c^2=\infty} = \frac{\displaystyle\sum_{k=1}^{2} \rho_k \int_{\Omega_k} |\nabla \Phi_k|^2 \, d\Omega_k}{\left\| B_\sigma^{-1/2} P_\Gamma (\rho_1 \Phi_1 - \rho_2 \Phi_2) \right\|_0^2}. \tag{198}$$

But in functional (195) this procedure is not correct. Here we must do the following: we divide (195) on c^2 and calculate the limit when $c^{-2} \longrightarrow 0$. We will have the functional

$$\lim_{c^{-2} \longrightarrow 0} c^{-2} F_2(\Phi_1, \Phi_2) = \frac{\rho_2 \displaystyle\int_{\Omega_2} |\Delta \Phi_2|^2 \, d\Omega_2}{\displaystyle\sum_{k=1}^{2} \rho_k \int_{\Omega_k} |\nabla \Phi_k|^2 \, d\Omega_k}. \tag{199}$$

It can be shown (see below) that functional (199) defines an asymptotic behavior of eigenvalues λc^{-2} corresponding to the so-called acoustic waves in studied hydrosystem.

3^0. Finally, if $\operatorname{mes} \Omega_1 \longrightarrow 0$ (and therefore $\operatorname{mes} \Gamma \longrightarrow 0$) then in a limit case a classical problem on oscillations of a barotropic gas in a region $\Omega_2 = \Omega$ arizes. Here we have variation relations

$$\frac{c^2 \displaystyle\int_{\Omega_2} |\, \nabla \Phi_2 \,|^2 \, d\Omega_2}{\displaystyle\int_{\Omega_2} |\, \Phi_2 \,|^2 \, d\Omega_2}, \qquad \frac{c^2 \displaystyle\int_{\Omega_2} |\, \Delta \Phi_2 \,|^2 \, d\Omega_2}{\displaystyle\int_{\Omega_2} |\, \nabla \Phi_2 \,|^2 \, d\Omega_2}, \tag{200}$$

corresponding to squared frequences of acoustic oscillations in $\Omega_2 = \Omega$.

5.4. **On surface and acoustic waves in the system „fluid – gas".** Here we will briefly consider some simple heuristic assertions connected with existence in the system „fluid – gas" of wave motions of two tipes.

Remark preliminary that if $c^2 = \infty$, i.e., the second fluid is incompressible, then we have in the system only surface waves. These waves are located in the vicinity of the equilibrium surface Γ (skin effect). Squared frequences of oscillations of these waves are consecutive minima of functional (198). From the other hand, the property of compressibility of the second fluid, as it is evident from physical considerations, must generate acoustic waves in the region Ω_2 fulfilled by a gas. In this, squared frequencies of oscillations of these types of waves are positive. They form a discrete spectrum with limit point at $+\infty$, i.e., both branches of eigenvalues are located on a positive semiaxis. Therefore it is very difficult to separate eigenvalues for these two types of waves. Here we must take into account not only eigenvalues but eigenfunctions of studied problem also.

Come back to problem (101), (102) and rewrite it in the form taking into account Lemma 4. We have

$$\rho_2 A\eta = \lambda\varepsilon(\eta + \rho_2 A^{-1/2}Q^*\zeta), \quad B_\sigma\zeta = \lambda(\rho_2 QA^{1/2}\eta + C\zeta), \tag{201}$$

$$Q := -P_\Gamma\gamma_2 A^{-1/2}, \quad Q^* = A^{1/2}T_2, \quad \varepsilon := c^{-2} > 0. \tag{202}$$

Consider solutions to problem (201), (202) as functions of a parameter $\varepsilon = c^{-2} > 0$. Remark that eigenvalues and eigenfunctions of the problem are continuous functions in ε when ε changes continuously on positive interval.

It is easily seen that solutions to problem (201), (202) are separated on two classes when $\varepsilon \longrightarrow +0$. For the first class we have $\lambda = \lambda(\varepsilon) = O(1)$ $(\varepsilon \longrightarrow +0)$, and for the second one $\lambda\varepsilon =: \mu = \mu(\varepsilon) = O(1)$ $(\varepsilon \longrightarrow +0)$. For the first class we have in the limit $\lambda = \lambda_0$, $\eta = \eta_0$, $\zeta = \zeta_0$, and for these elements relations

$$\rho_2 A\eta_0 = 0, \quad B_\sigma\zeta_0 = \lambda_0(\rho_2 QA^{1/2}\eta_0 + C\zeta_0), \tag{203}$$

are valid. Since $A \gg 0$, $B_\sigma \gg 0$, then it follows from (203) that $\eta_0 = 0$, $B_\sigma\zeta_0 = \lambda_0 C\zeta_0$. Then nontrivial solutions to system (203) have the form

$$\eta_0 = \eta_{0k} = 0, \quad \lambda_0 = \lambda_{0k}, \quad B_\sigma\zeta_{0k} = \lambda C\zeta_{0k}, \quad k = 1, 2, \ldots, \tag{204}$$

where λ_{0k} and ζ_{0k} are solutions to spectral problem (204). It corresponds to variation relation (198) and surface waves in the system of two capillary incompressible fluids. The problem has a discrete spectrum $\{\lambda_{0k}\}_{k=1}^\infty$ with limit point $+\infty$.

Thus, in problem (201), (202) there exist solutions (surface waves) of the form

$$\lambda = \lambda(\varepsilon) = \lambda_{0k} + o(1),\ \eta = \eta(\varepsilon) = o(1),\ \zeta = \zeta(\varepsilon) = \zeta_{0k} + o(1)\ (\varepsilon = c^{-2} \longrightarrow 0). \tag{205}$$

For the second class of solutions we consider the limit case $\mu(\varepsilon) = \lambda(\varepsilon)\varepsilon \longrightarrow \mu_0$ $(\varepsilon \longrightarrow +0)$, and from (201) we have the system of equations

$$\rho_2 A\eta = \mu_0(\rho_2\eta_0 + \rho_2 A^{-1/2}Q^*\zeta_0), \quad 0 = \mu_0(\rho_2 QA^{-1/2}\eta_0 + C\zeta_0). \tag{206}$$

It can be proved that this system (for $\mu_0 \neq 0$) has a discrete positive spectrum $\mu_0 = \mu_{0k}$, $k = 1, 2, \ldots$, with limit point $\mu = +\infty$, and numbers μ_{0k} can be finded as consecutive minima of variation relation (199). A physical sence of solutions of this form is the following: they are acoustic waves that are located not only in a gas (region Ω_2), across the surface Γ a fluid in a region Ω_1 also envolves in process of joint oscillations.

Thus, in the second case solutions to problem (201) have the form

$$\lambda(\varepsilon) = \mu_k(\varepsilon)\varepsilon^{-1} = \varepsilon^{-1}(\mu_{0k} + o(1)), \quad \eta(\varepsilon) = \eta_{0k} + o(1), \tag{207}$$

$$\zeta(\varepsilon) = \zeta_{0k} + o(1), \quad \varepsilon \longrightarrow +0, \quad k = 1, 2, \dots,$$

where μ_{0k} are eigenvalues of variation relation (199) and η_{0k} and ζ_{0k} are correspondent eigenelements (199) or system (206).

Comparing (205) and (207) we finally conclude that solutions to problem on oscillations of a system „fluid – gas" are asymptotically (as $c^2 \longrightarrow \infty$) separated on two classes of oscillations (surface and acoustic waves) for which frequencies have different asymptotic behavior.

6. On solvability of the initial boundary value problem.

Here we consider problems on unique solvability of the initial boundary value scalar problem (55) – (61) and the initial vector problem (8) – (15). The theorem on existence of strong solution to abstract hyperbolic equation in Hilbert space is the base for receiving these results.

6.1. **On transition to hyperbolic equation in Hilbert space.** Come back to scalar initial boundary value problem (55) – (61) for displacement potentials $\Phi_i(t, x)$, $i = 1, 2$. Spectral problem (63) – (68) correspond to it if solutions of homogeneous initial boundary value problem (55) – (61) have the form $\Phi_i(t, x) = e^{i\omega t}\Phi_i(x)$ (see (62)). Further, we used an operator approach for investigation of problem (63) – (68), and this approach led us to study of equations system (101) – (102).

We can use the same transforms in the initial boundary value problem (55) – (61) repeating the same way and considering that unknown functions are functions in variable t with values in corresponding Hilbert spaces. Then instead of (101) – (102) we come to Cauchy problem

$$\frac{d^2}{dt^2}(\rho_2\eta + \rho_2 T_2\zeta) + \rho_2 c^2 A\eta = \rho_2 F_2(t), \tag{208}$$

$$\frac{d^2}{dt^2}(-\rho_2 P_\Gamma \gamma_2 \eta + C\zeta) + B_\sigma\zeta = (\rho_1 F_1 - \rho_2 P_\Gamma F_2)\,|_\Gamma\,(t), \tag{209}$$

$$\eta(0) = \eta^0, \quad \eta'(0) = \eta^1, \quad \zeta(0) = \zeta^0, \quad \zeta'(0) = \zeta^1, \tag{210}$$

where we used the same notations and

$$\nabla F_2 = P_{2,G}\vec{f}, \quad \nabla F_1 = P_{1,h,S_1}\vec{f}, \quad \vec{f} = \vec{f}(t, x), \tag{211}$$

see (40), (33).

Further, we carry out the following formal transforms in problem (208) – (211). We use the substitutions

$$\eta = A^{-1/2}\widetilde{\eta}, \quad \zeta = C^{-1/2}\widetilde{\zeta}, \quad C = P_\Gamma(\rho_1 C_1 + \rho_2 C_2)P_\Gamma > 0. \tag{212}$$

Then, acting from the left by the operators $A^{1/2}$ in (208) and $C^{-1/2}$ in (209) (these steps will be justified), we will have

$$\frac{d^2}{dt^2}\left(\rho_2\widetilde{\eta} + \rho_2 Q^* C^{-1/2}\widetilde{\zeta}\right) + c^2\rho_2 A\widetilde{\eta} = \rho_2 A^{1/2} F_2(t), \tag{213}$$

$$\frac{d^2}{dt^2}\left(\rho_2 C^{-1/2}Q\widetilde{\eta}+\widetilde{\zeta}\right)+C^{-1/2}B_\sigma C^{-1/2}\widetilde{\zeta}=C^{-1/2}(\rho_1F_1-\rho_2P_\Gamma F_2)\mid_\Gamma(t), \tag{214}$$

$$\widetilde{\eta}(0)=\widetilde{\eta}^0=A^{1/2}\eta^0,\quad \widetilde{\eta}'(0)=A^{1/2}\eta^1,\quad \widetilde{\zeta}(0)=C^{1/2}\zeta^0,\quad \widetilde{\zeta}'(0)=C^{1/2}\zeta^1. \tag{215}$$

Remember that, by Lemma 4, the operators

$$Q:=-P_\Gamma(\gamma_\Gamma A^{-1/2}):L_{2,\Omega_2}\longrightarrow H_0=L_{2,\Gamma},\quad Q^*:=A^{1/2}T_2:H_0\longrightarrow L_{2,\Omega_2} \tag{216}$$

are mutual adjoint and compact.

Lemma 7. *The operators*

$$V:=C^{-1/2}Q:L_{2,\Omega_2}\longrightarrow H_0,\quad V^*:=QC^{-1/2}:H_0\longrightarrow L_{2,\Omega_2} \tag{217}$$

are mutual adjoint and bounded.

Proof. By Lemma 3, the operator $C^{-1/2}$ (after extention on H_0) act boundedly from H_0 onto $H_-=(H_\Gamma^{1/2})^*$; the operator T_2 is bounded from H_- into $H^1_{\Omega_2}$ (see Problem 2 and (89)); the operator $A^{1/2}$ is bounded from $H^1_{\Omega_2}=\mathcal{D}(A^{1/2})$ onto L_{2,Ω_2}. Therefore the operator $Q^*C^{-1/2}=A^{1/2}T_2C^{-1/2}$ is bounded from H_0 into L_{2,Ω_2}. Since V is adjoint to V^* then the operator $C^{-1/2}Q$ is bounded also. $\square$

Rewrite problem (213) – (215) in a vector-matrix form, i.e.,

$$\frac{d^2}{dt^2}(\mathcal{B}y)+\mathcal{A}y=\mathcal{F}(t),\quad y(0)=y^0,\quad y'(0)=y^1, \tag{218}$$

$$\mathcal{F}(t):=\Big(\rho_2A^{1/2}F_2(t);C^{-1/2}(\rho_1F_1-\rho_2P_\Gamma F_2)\mid_\Gamma(t)\Big)^t, \tag{219}$$

$$y=\begin{pmatrix}\widetilde{\eta}\\ \widetilde{\zeta}\end{pmatrix}\in\mathcal{H}(\Omega):=L_{2,\Omega_2}\oplus H_0,\quad y^0=\begin{pmatrix}\widetilde{\eta}^0\\ \widetilde{\zeta}^0\end{pmatrix},\quad y^1=\begin{pmatrix}\widetilde{\eta}^1\\ \widetilde{\zeta}^1\end{pmatrix}, \tag{220}$$

$$\mathcal{B}:=\begin{pmatrix}\rho_2I & \rho_2V^*\\ \rho_2V & I\end{pmatrix},\quad \mathcal{A}:=\begin{pmatrix}c^2\rho_2A & 0\\ 0 & C^{-1/2}B_\sigma C^{-1/2}\end{pmatrix}. \tag{221}$$

It follows from properties of the operators A, C^{-1} and B_σ that the operator $\mathcal{A}$, defined on the set

$$\mathcal{D}(\mathcal{A}):=\mathcal{D}(A)\oplus\mathcal{D}(C^{-1/2}B_\sigma C^{-1/2}),\quad \mathcal{D}(C^{-1/2}B_\sigma C^{-1/2})=\mathcal{R}(C^{1/2}B_\sigma^{-1}C^{1/2}), \tag{222}$$

is an unbounded selfadjoint positive definite operator acting in the space $\mathcal{H}(\Omega)$.

Lemma 8. *The operator $\mathcal{B}$ is a bounded selfadjoint and positive definite operator acting in $\mathcal{H}(\Omega)$.*

Proof. It follows from Lemma 7 that $\mathcal{B}$ is selfadjoint and bounded. Check that $\mathcal{B}$ is positive definite.

For any $y\in\mathcal{H}(\Omega)$ we have

$$(\mathcal{B}y,y)_{\mathcal{H}(\Omega)}=\Big(\rho_2\widetilde{\eta}+\rho_2V^*\widetilde{\zeta},\widetilde{\eta}\Big)_{\Omega_2}+\Big(\rho_2V\widetilde{\eta}+\widetilde{\zeta},\widetilde{\zeta}\Big)_0=$$

$$= \rho_2 \|\widetilde{\eta}\|^2_{\Omega_2} + 2\rho_2 \mathrm{Re}\left(V^*\widetilde{\zeta}, \widetilde{\eta}\right)_0 + \| \widetilde{\zeta} \|^2_0 .$$

Coming back to variables η and ζ by formulas (212), using relation (110), i.e.,

$$\| C^{1/2}\zeta \|^2_0 = \| \zeta \|^2_{H_-} = \rho_1 \int\limits_{\Omega_1} | \nabla\Phi_1 |^2 \; d\Omega_1 + \rho_2 \int\limits_{\Omega_2} | \nabla\Phi_{22} |^2 \; d\Omega_2 =$$

$$= \rho_1 \| T_1\zeta \|^2_{1,\Omega_1} + \rho_2 \| T_2\zeta \|^2_{1,\Omega_2}, \tag{223}$$

and solutions properties of auxiliary boundary value problems 1 – 3, we will have

$$(\mathcal{B}y, y)_{\mathcal{H}(\Omega)} = \rho_2 \| A^{1/2}\eta \|^2_{\Omega_2} + 2\rho_2 \left(A^{1/2}T_2C^{-1/2}C^{1/2}\zeta, A^{1/2}\eta\right)_{\Omega_2} + \rho_2 \| T_2\zeta \|^2_{1,\Omega_2} +$$

$$+ \rho_1 \| T_1\zeta \|^2_{1,\Omega_1} = \rho_2 \| A^{1/2}\eta \|^2_{\Omega_2} + 2\rho_2 \left(A^{1/2}\eta, A^{1/2}T_2\zeta\right)_{\Omega_2} + \rho_2 \| A^{1/2}T_2\zeta \|^2_{\Omega_2} +$$

$$+ \rho_1 \| T_1\zeta \|^2_{\Omega_1} \geqslant \rho_2 \left\{(1-\varepsilon) \| A^{1/2}\eta \|^2_{\Omega_2} + (1-\varepsilon^{-1}) \| A^{1/2}T_2\zeta \|^2_{\Omega_2}\right\} + \rho_1 \| T_1\zeta \|^2_{\Omega_1} =$$

$$= \rho_2 \left\{(1-\varepsilon) \| \widetilde{\eta} \|^2_{\Omega_2} + (1-\varepsilon^{-1}) \| T_2\zeta \|^2_{1,\Omega_2}\right\} + \rho_1 \| T_1\zeta \|^2_{1,\Omega_1}, \tag{224}$$

where ε is an arbitrary positive. We also used in (224) the property

$$\| A^{1/2}T_2\zeta \|_{\Omega_2} = \| T_2\zeta \|_{1,\Omega_2},$$

see Problem 3 and (96).

Coming back to Problems 1 and 2, observe that the following inequalities are valid for solutions to these problems:

$$\| T_2\zeta \|_{1,\Omega_2} \leqslant \| T_2 \| \cdot \| \zeta \|_- = \| T_2 \| \cdot \| T_1^{-1}T_1\zeta \|_- \leqslant \| T_2 \| \cdot \| T_1^{-1} \| \cdot \| T_1\zeta \|_{1,\Omega_1} =:$$

$$=: c^{-1} \| T_1\zeta \|_{1,\Omega_1}, \quad c > 0. \tag{225}$$

Therefore the right hand side in (224) can be evaluated from below, and we will have

$$(\mathcal{B}y, y)_{\mathcal{H}(\Omega)} \geqslant \rho_2(1-\varepsilon) \| \widetilde{\eta} \|^2_{\Omega_2} + [\rho_2(1-\varepsilon^{-1}) + \rho_1 c^2\alpha] \| T_2\zeta \|_{1,\Omega_2} +$$

$$+ \rho_1(1-\alpha) \| T_1\zeta \|^2_{1,\Omega_1}, \quad \alpha \in \mathbb{R}. \tag{226}$$

Take now parameters ε and α by such a way that the following relations will be valid:

$$0 < \varepsilon < 1, \quad 0 < \alpha < 1, \quad \rho_2(1-\varepsilon) = 1-\alpha = (1-\varepsilon^{-1}) + \rho_1 c^2 \rho_2^{-1} =: c_0 > 0. \tag{227}$$

It is easy to check that this system of equations has a unique solution, and then we have (for these ε and α) the inequality

$$(\mathcal{B}y, y)_{\mathcal{H}(\Omega)} \geqslant c_0 \left\{ \|\widetilde{\eta}\|^2_{\Omega_2} + \rho_2 \|T_2\zeta\|^2_{1,\Omega_2} + \rho_1 \|T_1\zeta\|^2_{1,\Omega_1} \right\} =$$

$$= c_0 \left\{ \|\widetilde{\eta}\|^2_{\Omega_2} + \left\|\widetilde{\zeta}\right\|^2_0 \right\} = c_0 \|y\|^2_{\mathcal{H}(\Omega)}, \quad c_0 > 0. \tag{228}$$

Here we also used relation (223). $\square$

Proved properties of the operators $\mathcal{B}$ and $\mathcal{A}$ show us that problem (218) is connected with Cauchy problem for hyperbolic equation in Hilbert space $\mathcal{H}(\Omega)$.

6.2. **On solvability of the initial boundary value problem for displacements potentials.** Further we use the following well know fact on solvability on Cauchy problem for equation of the form (218) (see, for instance, [9], pp. 60-63).

Theorem 8. *Let in problem* (218) *the operator* $\mathcal{B}$ *be bounded and positive definite and the operator* $\mathcal{A}$ *be selfadjoint (generally speaking, unbounded) positive definite. If the following conditions are valid, namely,*

$$y^0 \in \mathcal{D}(\mathcal{A}), \quad y^1 \in \mathcal{D}(\mathcal{A}^{1/2}), \quad \mathcal{F}(t) \in C^1(\mathbb{R}_+; \mathcal{H}(\Omega)), \tag{229}$$

then problem (218) *has a unique strong solution for* $t \geqslant 0$*, i.e., such a function* $y(t)$ *that*

$$y(t) \in \mathcal{D}(\mathcal{A}), \quad \forall\, t \in \mathbb{R}_+, \quad \mathcal{A}y(t) \in C(\mathbb{R}_+, \mathcal{H}(\Omega)),$$

$$y'(t) \in C(\mathbb{R}_+, \mathcal{D}(A^{1/2})), \quad y''(t) \in C(\mathbb{R}_+, \mathcal{H}(\Omega)),$$

and equation (218) *is valid for* $t \geqslant 0$ *and initial conditions are valid also.*

If instead of (229) *conditions*

$$y^0 \in \mathcal{D}(\mathcal{A}^{1/2}), \quad y^1 \in \mathcal{H}(\Omega), \quad \mathcal{F}(t) \in C(\mathbb{R}_+; \mathcal{H}(\Omega)) \tag{230}$$

hold, then problem (218) *has a generalized solution with continuous in* t *full energy. For this solution the law of full energy balance take place in the following form*

$$\frac{1}{2}\left\|\mathcal{B}^{1/2}\frac{dy}{dt}\right\|^2_{\mathcal{H}(\Omega)} + \frac{1}{2}\left\|\mathcal{A}^{1/2}y(t)\right\|^2_{\mathcal{H}(\Omega)} = \frac{1}{2}\left\|\mathcal{B}^{1/2}y^1\right\|^2_{\mathcal{H}(\Omega)} +$$

$$+\frac{1}{2}\left\|\mathcal{A}^{1/2}y^0\right\|^2_{\mathcal{H}(\Omega)} + \int_0^t \big(\mathcal{F}(s), y'(s)\big)_{\mathcal{H}(\Omega)}\, ds. \qquad \square \tag{231}$$

Remark 2. If the operator $\mathcal{A} = G^*G$ then instead of the second condition in (229) one can take condition $y^1 \in \mathcal{D}(G)$ and in (230) $\mathcal{D}(A^{1/2})$ must be changed by $\mathcal{D}(G)$. $\square$

On the base of Theorem 8 we will prove now some assertions on solvability of the initial boundary value problems on small motions of a system „fluid – gas".

Theorem 9. *Let in problem* (208) – (211) *the following conditions be satisfied:*

$$\eta^0 \in \mathcal{D}(A^{3/2}), \quad \eta^1 \in \mathcal{D}(A), \quad \zeta^0 \in \mathcal{D}(C^{-1/2}B_\sigma), \quad \zeta^1 \in \mathcal{D}(\mathcal{B}_\sigma^{1/2}), \tag{232}$$

$$\vec{f}(t,\cdot) \in C^1(\mathbb{R}_+; \vec{L}_2(\Omega)). \tag{233}$$

Then problem (208) – (211) *has (for* $t \geqslant 0$*) a strong solution with values in* $\mathcal{D}(A^{1/2}) \oplus \mathcal{D}(C^{-1/2})$*, i.e., such functions* $\eta(t)$ *and* $\zeta(t)$ *that the following properties are valid.*

1^0. $\eta(t) \in \mathcal{D}(A)$ *and* $A\eta(t) \in C(\mathbb{R}_+; \mathcal{D}(A^{1/2}))$*;*

2^0. $\eta(t) + T_2\zeta(t)$ *and* $\eta(t) \in C^2(\mathbb{R}_+; \mathcal{D}(A^{1/2}))$*;*

3^0. $\zeta(t) \in \mathcal{D}(B_\sigma)$ *and* $B_\sigma\zeta \in C(\mathbb{R}_+; H_\Gamma^{1/2})$*;*

4^0. $-\rho_2 P_\Gamma \gamma_2 \eta(t) + P_\Gamma(\rho_1 C_1 + \rho_2 C_2)\zeta(t)$ *and* $P_\Gamma(\rho_1 C_1 + \rho_2 C_2)P_\Gamma\zeta(t) \in C^2(\mathbb{R}_+; H_\Gamma^{1/2})$*;*

5^0. *equations* (208) *and* (209) *hold, any term in* (208) *is a function in* t *with values in* $\mathcal{D}(A^{1/2}) = H^1_{\Omega_2}$ *and any term in* (209) *is a function in* t *with values in* $\mathcal{D}(C^{-1/2}) = H^{1/2}_\Gamma$*;*
6^0. *initial conditions* (210) *hold.*

If the following conditions are valid,

$$\eta^0 \in \mathcal{D}(A), \quad \eta^1 \in \mathcal{D}(A^{1/2}), \quad \zeta^0 \in \mathcal{D}(B^{1/2}_\sigma), \quad \zeta^1 \in (H^{1/2}_\Gamma)^* = H_-, \tag{234}$$

$$\vec{f}(t,x) \in C(\mathbb{R}_+; \vec{L}_2(\Omega)), \tag{235}$$

then problem (208) – (209) *has a unique solution with continuous full energy, i.e., such functions that the law of full energy balance* (231) *holds and any term in this relation is a continuous function in* $t \in \mathbb{R}_+$*.*

Proof. If conditions (232), (233) hold then in problem (218) – (221) (with taking into account (212)) we have

$$\widetilde{\eta}^0 \in \mathcal{D}(A), \quad \widetilde{\eta}^1 \in \mathcal{D}(A^{1/2}), \quad \widetilde{\zeta}^0 \in \mathcal{D}(C^{-1/2} B_\sigma C^{-1/2}), \quad \widetilde{\zeta}^1 \in \mathcal{D}(B^{1/2}_\sigma C^{-1/2}). \tag{236}$$

Further, if $\vec{f}(t,x) \in C^1(\mathbb{R}_+; \vec{L}_2(\Omega))$ then $\nabla F_2 = P_{2,G}\vec{f} \in C^1(\mathbb{R}_+; \vec{G}(\Omega_2))$, $F_2 \in C^1(\mathbb{R}_+; H^1_{\Omega_2})$ and therefore $\rho_2 A^{1/2} F_2 \in C^1(\mathbb{R}_+; L_{2,\Omega_2})$ since $H^1_{\Omega_2} = \mathcal{D}(A^{1/2})$. Next, $\nabla F_1 = P_{1,h,S_1}\vec{f} \in C^1(\mathbb{R}_+; \vec{G}_{h,S_1}(\Omega_1))$ and $F_1 \in C^1(\mathbb{R}_+; H^1_{h,S_1}(\Omega_1))$. Therefore

$$(\rho_1 F_1 - \rho_2 P_\Gamma F_2)\,|_\Gamma \in C^1(\mathbb{R}_+; H^{1/2}_\Gamma) = C^1(\mathbb{R}_+; \mathcal{D}(C^{-1/2})). \tag{237}$$

Thus, if conditions (232), (233) hold then conditions (229) are valid (see also Remark 2, problem (218)). Therefore, by Theorem 8, problem (218) has a unique strong solution on $\mathbb{R}_+$. It means that equations (213) and (214) are valid and any term in these equations is a continuous in t function with values in the spaces L_{2,Ω_2} and $H_0 = L_{2,\Gamma}$, respectively.

Come back from (213), (214) to system (208), (209) using inverse substitutions (212). Acting from the left by the (bounded) operator $A^{-1/2}$ to (213) and by the (bounded) operator $C^{1/2}$ to (214), we conclude that system of equations (208), (209) hold, and in (208) any term is a function in t with values in $\mathcal{D}(A^{1/2}) = H^1_{\Omega_2}$ and in (209) any term is a function in t with values in $\mathcal{D}(C^{-1/2}) = H^{1/2}_\Gamma$. In other words, problem (208), (209) has a strong solution $(\eta(t); \zeta(t))^t$ with values in $\mathcal{D}(A^{1/2}) \oplus \mathcal{D}(C^{-1/2})$. Note else that properties

$$\eta(t) \in C^2(\mathbb{R}_+; \mathcal{D}(A^{1/2})), \quad P_\Gamma(\rho_1 C_1 + \rho_2 C_2) P_\Gamma \zeta(t) \in C^2(\mathbb{R}_+; H^{1/2}_\Gamma) \tag{238}$$

follow from the fact that $\dfrac{d^2}{dt^2}(\mathcal{B}y(t)) \in C(\mathbb{R}_+; \mathcal{H}(\Omega))$ and invertibility of the operator $\mathcal{B}$ (Lemma 8).

Proof of existence of the generalized solution to problem (208) – (211) has the same way and therefore it is absent here. $\square$

Taking into account Theorem 9 we will prove now the theorem on strong solvability of the initial boundary value problem (55) – (61) for displacement potentials.

Theorem 10. *Let in problem* (55) – (61) *the following conditions be satisfied:*

$$\Phi_1^0 \in H^1_{h,S_1}(\Omega_1), \quad \Phi_2^0 = \Phi_{21}^0 + \Phi_{22}^0, \tag{239}$$

$$\Phi_{22}^0 \in H^1_{h,S_2}(\Omega_2), \quad \frac{\partial \Phi_{22}^0}{\partial n}\,|_\Gamma = \frac{\partial \Phi_1^0}{\partial n}\,|_\Gamma =: \zeta^0 \in \mathcal{D}(B_\sigma) \cap H_\Gamma^{3/2}, \tag{240}$$

$$\Delta \Phi_{21}^0 \in H^1_{\Omega_2}, \quad \Phi_1^1 \in H^1_{1,S_1}(\Omega_1), \quad \Phi_2^1 = \Phi_{21}^1 + \Phi_{22}^1, \tag{241}$$

$$\Phi_{22}^1 \in H^1_{h,S_2}(\Omega_2), \quad \frac{\partial \Phi_{22}^1}{\partial n}\,|_\Gamma = \frac{\partial \Phi_1^1}{\partial n}\,|_\Gamma =: \zeta^1 \in \mathcal{D}(B_\sigma^{1/2}) = H_\Gamma^1, \quad \Phi_{21}^1 \in \mathcal{D}(A), \tag{242}$$

$$\vec{f}(t,\cdot) \in C^1(\mathbb{R}_+; \vec{L}_2(\Omega)). \tag{243}$$

Then problem (55) – (61) *has a unique strong (in t) solutions with values in the space*

$$\mathcal{H}^1(\Omega;\rho) := \Big\{(\Phi_2;\Phi_1):\ \Phi_1 \in H^1_{h,S_1}(\Omega_1), \quad \Phi_2 = \Phi_{21} + \Phi_{22},$$

$$\Phi_{22} \in H^1_{h,S_2}(\Omega_2) \subset H^1_{\Omega_2}, \quad \Phi_{21} \in H^1_{\Omega_2}, \quad \Delta\Phi_{21} \in (H^1_{\Omega_2})^*, \quad \frac{\partial \Phi_{21}}{\partial n} = 0\ (on\ S_2),$$

$$\frac{\partial \Phi_{21}}{\partial n} = 0\ (on\ \Gamma), \quad \frac{\partial \Phi_{22}}{\partial n} = \frac{\partial \Phi_2}{\partial n} = \frac{\partial \Phi_1}{\partial n} \in (H_\Gamma^{1/2})^*\ (on\ \Gamma)\Big\}, \tag{244}$$

i.e., such functions $\Phi_1(t,x)$ and $\Phi_2(t,x)$ that the following properties are valid.

1^0. $\Phi_2(t,x) = \Phi_{21}(t,x) + \Phi_{22}(t.x)$ *with* $\Delta\Phi_{21}(t,x) \in C(\mathbb{R}_+; H^1_{\Omega_2})$ *and* $\Phi_{22}(t,x) \in C(\mathbb{R}_+; H^1_{h,S_2}(\Omega_2))$;

2^0. $\Phi_2(t,x) \in C^2(\mathbb{R}_+; H^1_{\Omega_2})$;

3^0. *for any $t \geqslant 0$ equation* (56) *holds and any term in it is a continuous function in t with values in $H^1_{\Omega_2}$;*

4^0. $\Phi_1(t,x) \in C(\mathbb{R}_+; H^1_{h,S_1}(\Omega_1))$ *and* $\dfrac{\partial \Phi_1}{\partial n}\,|_\Gamma = \dfrac{\partial \Phi_2}{\partial n}\,|_\Gamma = \dfrac{\partial \Phi_{22}}{\partial n}\,|_\Gamma \in C(\mathbb{R}_+;\ \mathcal{D}(C^{-1/2}B_\sigma))$;

5^0. $\Phi_1(t,x)$ *and* $P_\Gamma \Phi_2(t,x)$, $x \in \Gamma$, *belong to the space* $C^2(\mathbb{R}_+; H_\Gamma^{1/2})$ *and equation* (59) *holds for any $t \geqslant 0$;*

6^0. *initial conditions* (60), (61) *hold, i.e.,*

$$\Phi_i(0,x) = \Phi_i^0(x), \quad \frac{\partial}{\partial t}\Phi_i(0,x) = \Phi_i^1(x), \quad x \in \Omega_i,\ i = 1,2. \tag{245}$$

Proof. If conditions (239) – (243) hold then, as it is easy to see, initial conditions (232), (233) are valid in problem (208) – (211). Indeed, according to Subsection 3.2 (see (98)), we have

$$\Phi_2(t,x) = \Phi_{21}(t,x) + \Phi_{22}(t,x), \quad \Phi_{22}(t,x) = T_2\zeta(t,x) \quad \text{(Problem 2)}, \tag{246}$$

$$\Phi_{21}(t,x) =: \eta(t,x) \quad \text{(see (100))}, \quad \Phi_1(t,x) = T_1\zeta(t,x) \quad \text{(Problem 1)}. \tag{247}$$

It follows from (246), (247) and (239) – (243) that all conditions of Theorem 9 are fulfilled. In particular, $\eta(0) = \eta^0 \in \mathcal{D}(A^{3/2})$ since $A\eta^0 = -\Delta\Phi_{21}^0 \in H^1_{\Omega_2} = \mathcal{D}(A^{1/2})$, $\zeta^0 \in \mathcal{D}(C^{-1/2}B_\sigma)$, $\zeta^1 \in \mathcal{D}(B_\sigma^{1/2})$, $\eta^1 = \Phi_{21}^1 \in \mathcal{D}(A)$ and (243) is the same as (233).

It follows from Theorem 9 that problem (208) – (211) has a unique strong solution with values in $\mathcal{D}(A^{1/2}) \oplus \mathcal{D}(C^{-1/2})$. Then $\eta(t) = \Phi_{21}(t,x) \in C(\mathbb{R}_+; \mathcal{D}(A^{3/2}))$ and $A\eta(t) = -\Delta\Phi_{21}(t,x) \in C(\mathbb{R}_+; H^1_{\Omega_2})$. Next, since $\zeta(t) \in C(\mathbb{R}_+; \mathcal{D}(C^{-1/2}B_\sigma))$ then, using solutions properties of auxiliary Problems 1 and 2, we conclude that

$$\Phi_{22}(t,x) = T_2\zeta(t) \in C(\mathbb{R}_+; H^1_{h,S_2}(\Omega_2), \quad \Phi_1(t,x) = T_1\zeta(t) \in C(\mathbb{R}_+; H^1_{h,S_1}(\Omega_1).$$

Therefore

$$\Phi_2(t,x) = \Phi_{21}(t,x) + \Phi_{22}(t,x) \in C(\mathbb{R}_+; H^1_{\Omega_2}),$$
$$\Delta\Phi_2(t,x) = \Delta\Phi_{21}(t,x) \in C(\mathbb{R}_+; H^1_{\Omega_2}).$$

From equation (208) and Theorem 8 it follows also that

$$\Phi_2(t,x) = \eta(t) + T_2\zeta(t) = \Phi_{21}(t,x) + \Phi_2(t,x) \in C^2(\mathbb{R}_+; H^1_{\Omega_2}), \tag{248}$$

and from equation (209) we see that

$$-\rho_2 P_\Gamma \gamma_2 \eta(t) + P_\Gamma(\rho_1 C_1 + \rho_2 C_2) P_\Gamma \zeta(t) + P_\Gamma(\rho_1\Phi_1 - \rho_2\Phi_2) \in C^2(\mathbb{R}_+; H^{1/2}_\Gamma),$$
$$P_\Gamma(\rho_1\Phi_1 - \rho_2\Phi_2) \in C^2(\mathbb{R}_+; H^{1/2}_\Gamma) \tag{249}$$

(see (238)). It follows from (248) and embedding theorem that $\Phi_2 \mid_\Gamma \in C^2(\mathbb{R}_+; H^{1/2}_\Gamma)$. Then from (249) we conclude that $\Phi_1 \mid_\Gamma \in C^2(\mathbb{R}_+; H^{1/2}_\Gamma)$. Besides, we know that

$$\frac{\partial\Phi_1}{\partial n} \mid_\Gamma = \frac{\partial\Phi_2}{\partial n} \mid_\Gamma = \frac{\partial\Phi_{22}}{\partial n} \mid_\Gamma =: \zeta(t) \in C(\mathbb{R}_+; \mathcal{D}(C^{-1/2}B_\sigma)). \tag{250}$$

Note, at last, that introduced functions $\Phi_1(t,x)$ and $\Phi_2(t,x)$ are solutions to equation (55) and (56) (all terms in (56) are continuous in t functions with values in $H^1_{\Omega_2}$), kinematic condition (59) (all terms are from $C(\mathbb{R}_+; H^{1/2}_\Gamma)$) and boundary conditions (57). Besides, initial conditions to problem (55) – (61) are fulfilled. □

6.3. **On solvability of the initial boundary value vector problem.** Above proved theorems give us opportunity to prove theorem on unique solvability of an initial boundary value vector problem (8) – (15) on small motions of a hydrosystem „fluid – gas".

Theorem 11. *Let in problem* (8) – (15) *the following conditions be fulfilled,*

$$\vec{w}_1^0 = \nabla\Phi_1^0 + P_{1,0}\vec{w}_1^0 \in \vec{L}_2(\Omega_1), \quad \nabla\Phi_1^0 \in \vec{G}_{h,S_1}(\Omega_1), \tag{251}$$
$$\vec{w}_1^1 = \nabla\Phi_1^1 + P_{1,0}\vec{w}_1^1 \in \vec{L}_2(\Omega_1), \quad \nabla\Phi_1^1 \in \vec{G}_{h,S_1}(\Omega_1), \tag{252}$$
$$\vec{w}_2^0 = \nabla\Phi_2^0 + P_{2,0}\vec{w}_2^0 \in \vec{L}_2(\Omega_2), \quad \nabla\Phi_2^0 \in \vec{G}(\Omega_2), \tag{253}$$
$$\vec{w}_2^1 = \nabla\Phi_2^1 + P_{2,0}\vec{w}_2^1 \in \vec{L}_2(\Omega_2), \quad \nabla\Phi_2^1 \in \vec{G}(\Omega_2), \tag{254}$$
$$\vec{f} \in C^1(\mathbb{R}_+; \vec{L}_2(\Omega)), \tag{255}$$

where initial potentials Φ_i^0, Φ_i^1, $i = 1,2$, have, as in Theorem 10, properties (239) – (242).

Then problem (8) – (15) *has a unique strong solution with values in the space $\vec{L}_2(\Omega) := \vec{L}_2(\Omega_1) \oplus \vec{L}_2(\Omega_2)$. Namely, there exist functions $\vec{w}_i(t,x)$, $p_i(t,x)$, $i = 1,2$, such that* *equations* (8) *and* (9) *hold; all terms in the first equation* (8) *are functions in* t *with values in* $\vec{L}_2(\Omega_1)$*; all terms in the first equation* (9) *are functions to in* t *with values in* $\vec{L}_2(\Omega_2)$*; all terms in the second equation* (9) *are functions in* t *with values in* $H^1_{\Omega_1}$.

Further, kinematic condition (11) *holds in the space* $C(\mathbb{R}_+;\mathcal{D}(C^{-1/2}B_\sigma))$ *(i.e.,* $B_\sigma\zeta\in C(\mathbb{R}_+;H^{1/2}_\Gamma)$*), dynamic condition* (12) *hods in* $C(\mathbb{R}_+;H^{1/2}_\Gamma)$*, and initial conditions* (15) *are fulfilled.*

Proof. 1). It follows from (255) and (251), (252) that problem (31) has a unique solution

$$\vec{v}_1=P_{1,0}\vec{w}_1^0+\int\limits_0^t\left(P_{1,0}\vec{w}_1^1+\int\limits_0^s P_{1,0}\vec{f}(\xi)\,d\xi\right)ds=$$

$$=P_{1,0}\left(\vec{w}_1^0+t\vec{w}_1^1+\int\limits_0^t ds\int\limits_0^s \vec{f}(\xi)\,d\xi\right)\in C^3(\mathbb{R}_+;\vec{J}_0(\Omega_1)) \tag{256}$$

and, by (32),

$$\nabla\varphi_1:=\rho_1P_{1,0,\Gamma}\vec{f}\in C^1(\mathbb{R}_+;\vec{G}_{0,\Gamma}(\Omega_1)). \tag{257}$$

2). Similarly, from (41) we have

$$\vec{v}_2=P_{2,0}\left(\vec{w}_2^0+t\vec{w}_2^1+\int\limits_0^t ds\int\limits_0^s \vec{f}(\xi)\,d\xi\right)\in C^3(\mathbb{R}_+;\vec{J}_0(\Omega_2)). \tag{258}$$

3). Since initial potentials Φ_i^0, Φ_i^1, $i=1,2$, have properties (239) – (242) (and by (255)), then assertions of Theorem 10 hold. In particular, $\Phi_2(t,x)\in C^2(\mathbb{R}_+;H^1_{\Omega_2})$. Therefore,

$$\nabla p_2:=\rho_2\left(\frac{\partial^2}{\partial t^2}\nabla\Phi_2-\nabla F_2\right)\in C(\mathbb{R}_+;\vec{G}(\Omega_2)), \tag{259}$$

and then equation (40) holds and any term in it is a function in t with values in $C(\mathbb{R}_+;\vec{G}(\Omega_2))$.

4). It follows from this property and equation (59) that

$$\Phi_1\,|_\Gamma=:\varphi_1\in C^2(\mathbb{R}_+;H^{1/2}_\Gamma). \tag{260}$$

Consider now auxiliary Zaremba problem

$$\Delta\Phi_1=0\ (\text{in }\Omega_1),\quad \frac{\partial\Phi_1}{\partial n}=0\ (\text{on }S),\quad \Phi_1=\varphi_1\ (\text{on }\Gamma). \tag{261}$$

It is known (see, for instance, [9], p. 107), that problem (261) has a unique generalized solution $\Phi_1\in H^1_{h,S_1}(\Omega_1)$ if and only if $\varphi_1\in H^{1/2}_\Gamma$. Moreover, if condition (260) holds then

$$\Phi_1\in C^2(\mathbb{R}_+;H^1_{\Omega_1}). \tag{262}$$

5). Taking into account (262) and property $\nabla F_1 = P_{1,h,S_1}\vec{f} \in C^1(\mathbb{R}_+; \vec{G}_{h,S_1}(\Omega_1))$, introduce, by (33),

$$\widetilde{\nabla p_1} := \rho_1 \nabla F_1 - \rho_1 \frac{\partial^2}{\partial t^2}\nabla\Phi_1 \in C(\mathbb{R}_+; \vec{G}_{h,S_1}(\Omega_1)). \tag{263}$$

Then all equations (31) – (33) are valid for $t \geqslant 0$ and all terms in these equations are continuous functions in t with values in $\vec{J}_0(\Omega_1)$, $\vec{G}_{0,\Gamma}(\Omega_1)$ and $\vec{G}_{h,S_1}(\Omega_1)$, respectively. From this it follows that functions

$$\vec{w}_1 = \vec{v}_1 + \nabla\Phi_1, \quad \nabla p_1 = \widetilde{\nabla p_1} + \nabla\varphi_1 \tag{264}$$

(see (27) – (30)) are solutions to equation (8) and all terms in it are functions in t with values in $\vec{L}_2(\Omega_1)$. Besides, the second condition in (8) is also valid.

6). Introduce

$$\vec{w}_2 = \vec{v}_2 + \nabla\Phi_2 \tag{265}$$

and use properties (258), (259). Then we see that the first equation (9) holds and each term is a function in t with values in $\vec{L}_2(\Omega_2)$. Besides, the second equation (9) is valid with terms from $C(\mathbb{R}_+; H^1_{\Omega_2})$.

7). It follows from (250) that

$$\zeta := (\vec{w}_1 \cdot \vec{n})_\Gamma = (\vec{w}_2 \cdot \vec{n})_\Gamma = \left(\frac{\partial\Phi_1}{\partial n}\right)_\Gamma = \left(\frac{\partial\Phi_2}{\partial n}\right)_\Gamma \in C(\mathbb{R}_+; \mathcal{D}(C^{-1/2}B_\sigma)). \tag{266}$$

Then $B_\sigma\zeta \in C(\mathbb{R}_+; \mathcal{D}(C^{-1/2})) = C(\mathbb{R}_+; H^{1/2}_\Gamma)$ and, by (263) – (265), (45), (59),

$$(p_1 - p_2)_\Gamma = \mathcal{L}_\sigma\zeta \in C(\mathbb{R}_+; H^{1/2}_\Gamma). \tag{267}$$

8). Thus, all equations (8) – (15) hold. In particular, $\zeta \in \mathcal{D}(B_\sigma)$ and therefore condition (14) is valid; initial conditions (15) are also fulfilled by (256), (60), (61).

This proves the theorem. $\square$

As a corollary of the Theorem 11 we have the following result.

Theorem 12. *If conditions of Theorem 11 are fulfilled then the law on full energy balance in the form* (20) *is valid, and this energy is a continuous function in* $t \geqslant 0$.

If the following conditions,

$$(\vec{w}_2; \vec{w}_1) \in \vec{G}(\Omega), \quad \operatorname{div}\vec{w}_2^0 \in L_2(\Omega_2),$$

$$\vec{w}_1^0 \cdot \vec{n} = \vec{w}_2^0 \cdot \vec{n} \in \mathcal{D}(B_\sigma^{1/2}) = H^1_\Gamma, \quad \vec{f}(t,x) \in C(\mathbb{R}_+; \vec{L}_2(\Omega)), \tag{268}$$

hold, then problem (8) – (15) *has a generalized solution with continuous full energy, and the law* (20) *also holds for this solution.*

Proof. 1). If conditions of Theorem 11 are fulfilled, then we can repeat the same transforms as in Subsection 2.2 and receive the law of full energy balance in the form (20).

2). Proof of the second part of the theorem follows from the fact that generalized solutions with continuous full energy can be received on any segment $[0,T]$ by limit transition from initial conditions (251) – (255), corresponding to strong solutions, to initial conditions (268), corresponding to generalized one. We remark only that the second condition (267) is equivalent to condition $p_2(x,0) \in L_{2,\Omega_2}$. □

6.4. **Fourier series.** On the base of Theorems 10 and 11 on existence of strong solutions initial boundary value problems (55) – (61) and (8) – (15) and on the base of Theorems 6, 7 on basis properties of the system of eigenfunctions to spectral problems (63) – (68) and (187) – (189) we can receive Fourier series for solutions to problem (55) – (61).

Remember, that eigenfunctions $\Phi_k := (\Phi_{2k}; \Phi_{1k})$, $k = 1, 2, \ldots$, to problem (63) – (68) are solutions to the following spectral problem:

$$\Delta\Phi_{1k} = 0 \ (\text{in } \Omega_1), \quad -\Delta\Phi_{2k} = \lambda_k c^{-2}\Phi_{2k} \ (\text{in } \Omega_2), \tag{269}$$

$$\frac{\partial\Phi_{1k}}{\partial n} = 0 \ (\text{on } S_1), \quad \frac{\partial\Phi_{2k}}{\partial n} = 0 \ (\text{on } S_2), \quad \frac{\partial\Phi_{1k}}{\partial n} = \frac{\partial\Phi_{2k}}{\partial n} =: \zeta_k \ (\text{on } \Gamma), \tag{270}$$

$$\zeta_k = \lambda_k B_\sigma^{-1} P_\Gamma(\rho_1\Phi_{1k} - \rho_2\Phi_{2k}) \ (\text{on } \Gamma), \quad \int_\Gamma \zeta_k \, d\Gamma = 0, \quad \int_{\Omega_2} \Phi_{2k} \, d\Omega_2 = 0. \tag{271}$$

These functions form an orthogonal basis in $\mathcal{H}_1(\Omega; \rho)$ (with squared norm (176)) and have the following conditions on orthonormality:

$$\sum_{k=1}^{2} \rho_m \int_{\Omega_m} \nabla\Phi_{mk} \cdot \nabla\Phi_{ml} \, d\Omega_m = \delta_{kl}, \tag{272}$$

$$\rho_2 c^{-2} \int_{\Omega_2} \Phi_{2k} \cdot \Phi_{2l} \, d\Omega_2 + \int_\Gamma B_\sigma^{-1}[P_\Gamma(\rho_1\Phi_{1k} - \rho_2\Phi_{2k})] \cdot [P_\Gamma(\rho_1\Phi_{1l} - \rho_2\Phi_{2l})] \, d\Gamma = \lambda_k^{-1}\delta_{kl}. \tag{273}$$

Consider for simplicity the initial boundary value problem (55) – (61) for the case of free motions, i.e., $\vec{f}(t,x) \equiv \vec{0}$. Then $F_1(t,x) \equiv 0$, $F_2(t,x) \equiv 0$. Represent solution $\Phi = (\Phi_2(t,x); \Phi_1(t,x))^t$ to problem (55) – (61) in the form

$$\begin{pmatrix} \Phi_2(t,x) \\ \Phi_1(t,x) \end{pmatrix} = \sum_{k=1}^{\infty} c_k(t) \begin{pmatrix} \Phi_{2k}(x) \\ \Phi_{1k}(x) \end{pmatrix}, \tag{274}$$

where $c_k(t)$ are anknown functions and $\Phi_k = (\Phi_{2k}; \Phi_{1k})^t$, $k = 1, 2, \ldots$, are solutions to spectral problem (269) – (271) with properties (272), (273).

We put functions $\Phi_2(t,x)$ and $\Phi_1(t,x)$ from (274) into equations (55), (56) and boundary conditions (57), (58). Further, we multiply the first relation on $-\rho_1\Phi_{1l}$ and integrate over Ω_1, the second one on $\rho_2\Phi_{2l}$ and integrate over Ω_2. Finally, we act by B_σ^{-1} from the left in (59), multiply on $(\rho_1\Phi_{1k} - \rho_2\Phi_{2k})$ and integrate over Γ. Using also boundary conditions (270), (271), we have

$$0 = \sum_{k=1}^{\infty} c_k(t) \left(\rho_1 \int\limits_{\Omega_1} \nabla\Phi_{1k} \cdot \nabla\Phi_{1l}\, d\Omega_1 - \rho_1 \int\limits_{\Gamma} \frac{\partial \Phi_{1k}}{\partial n} \Phi_{1l}\, d\Gamma \right),$$

$$0 = \sum_{k=1}^{\infty} \left[c_k''(t) c^{-2} \rho_2 \int\limits_{\Omega_2} \Phi_{2k} \cdot \Phi_{2l}\, d\Omega_2 + c_k(t)\rho_2 \left(\int\limits_{\Omega_2} \nabla\Phi_{2k} \cdot \nabla\Phi_{2l}\, d\Omega_2 + \int\limits_{\Gamma} \frac{\partial \Phi_{2k}}{\partial n} \Phi_{2l}\, d\Gamma \right) \right],$$

$$0 = \sum_{k=1}^{\infty} c_k''(t) \int\limits_{\Gamma} B_\sigma^{-1} [P_\Gamma(\rho_1\Phi_{1k} - \rho_2\Phi_{2k})][P_\Gamma(\rho_1\Phi_{1l} - \rho_2\Phi_{2l})]\, d\Gamma +$$

$$+ \sum_{k=1}^{\infty} c_k''(t) \int\limits_{\Gamma} \frac{\partial \Phi_{1k}}{\partial n} (\rho_1\Phi_{1l} - \rho_2\Phi_{2l})\, d\Gamma.$$

Adding the left and the right parts of these relations we receive the equality

$$0 = \sum_{k=1}^{\infty} c_k(t) \left(\rho_1 \int\limits_{\Omega_1} \nabla\Phi_{1k} \cdot \nabla\Phi_{1l}\, d\Omega_1 + \rho_2 \int\limits_{\Omega_2} \nabla\Phi_{2k} \cdot \nabla\Phi_{2l}\, d\Omega_2 \right) +$$

$$+ \sum_{k=1}^{\infty} c_k''(t) \left(\rho_2 c^{-2} \int\limits_{\Omega_2} \Phi_{2k}\Phi_{2l}\, d\Omega_2 + \int\limits_{\Gamma} B_\sigma^{-1} [P_\Gamma(\rho_1\Phi_{1k} - \rho_2\Phi_{2k})][P_\Gamma(\rho_1\Phi_{1l} - \rho_2\Phi_{2l})]\, d\Gamma \right),$$

that with taking into account (272), (273) gives the equations

$$c_l(t) + \lambda_l^{-1} c_l''(t) = 0, \quad l = 1, 2, \ldots.$$

From this it follows that

$$c_k(t) = c_{k0}\cos(\omega_k t) + c_{k1}\sin(\omega_k t), \quad \omega_k = \sqrt{\lambda_k}, \quad k = 1, 2, \ldots, \tag{275}$$

and therefore the formal solution to problem (55) – (59) has the form

$$\begin{pmatrix} \Phi_2(t,x) \\ \Phi_1(t,x) \end{pmatrix} = \sum_{k=1}^{\infty} (c_{k0}\cos(\omega_k t) + c_{k1}\sin(\omega_k t)) \begin{pmatrix} \Phi_{2k}(x) \\ \Phi_{1k}(x) \end{pmatrix}. \tag{276}$$

One can find coefficients $\{c_{k0}\}_{k=1}^{\infty}$ and $\{c_{k1}\}_{k=1}^{\infty}$ using the initial conditions (60), (61):

$$\Phi_i(0,x) = \Phi_i^0(x), \quad \frac{\partial}{\partial t}\Phi_i(0,x) = \Phi_i^1(x), \quad i = 1, 2. \tag{277}$$

We have

$$\begin{pmatrix} \Phi_2^0(x) \\ \Phi_1^0(x) \end{pmatrix} = \sum_{k=1}^{\infty} \alpha_k \begin{pmatrix} \Phi_{2k}(x) \\ \Phi_{1k}(x) \end{pmatrix}, \quad \begin{pmatrix} \Phi_2^1(x) \\ \Phi_1^1(x) \end{pmatrix} = \sum_{k=1}^{\infty} \beta_k \begin{pmatrix} \Phi_{2k}(x) \\ \Phi_{1k}(x) \end{pmatrix}, \tag{278}$$

and, by (272),

$$\alpha_k = \sum_{j=1}^{2} \rho_j \int\limits_{\Omega_j} \nabla\Phi_j^0 \cdot \nabla\Phi_{1k}\, d\Omega_j, \quad \beta_k = \sum_{j=1}^{2} \rho_j \int\limits_{\Omega_j} \nabla\Phi_j^1 \cdot \nabla\Phi_{1k}\, d\Omega_j. \tag{279}$$

Using (278), (279) and initial conditions (277), we have finally

$$\begin{pmatrix} \Phi_2(t,x) \\ \Phi_1(t,x) \end{pmatrix} = \sum_{k=1}^{\infty} \left(\alpha_k \cos(\omega_k t) + \beta_k \omega_k^{-1} \sin(\omega_k t)\right) \begin{pmatrix} \Phi_{2k}(x) \\ \Phi_{1k}(x) \end{pmatrix}. \tag{280}$$

This solution is a strong one with values in $\mathcal{H}^1(\Omega;\rho)$ if initial functions (277) have properties as in Theorem 10, i.e., properties (239) – (242).

On the base of the above proved results and (280) one can represent solution to the initial boundary value vector problem (8) – (15).

6.5. **On sufficient condition for instability on small motions of the system „fluid – gas".** Remember that up to this moment we used an assumption on statical stability of the system „fluid – gas" (see (69)), i.e., the operator B_σ is positive definite. Consider now the case when B_σ is only bounded from below and $\gamma < 0$ (Lemma 1). Then, as in Lemma 2 and assertions below, the operator B_σ has a discrete spectrum $\{\lambda_k(B_\sigma)\}_{k=1}^{\infty} \subset \mathbb{R}$. But now its eigenvalues have the following properties (with taking into account its multiplicities)

$$-\infty < \gamma \leqslant \lambda_1(B_\sigma) \leqslant \ldots \leqslant \lambda_\varkappa(B_\sigma) < 0 = \lambda_{\varkappa+1}(B_\sigma) = \ldots = \lambda_{\varkappa+q}(B_\sigma) <$$

$$< \lambda_{\varkappa+q+1}(B_\sigma) \leqslant \ldots \leqslant \lambda_k(B_\sigma) \leqslant \ldots. \tag{281}$$

Consider (in assumption, that $\varkappa \geqslant 1$, $q \geqslant 0$ in (281)) solutions to homogeneous problem (218) in the form of the oscillations:

$$y(t) = e^{i\omega t} y, \quad y \in \mathcal{D}(\mathcal{A}). \tag{282}$$

Then for amplitude elements y we have the spectral problem

$$\mathcal{A}y = \lambda \mathcal{B}y, \quad y \in \mathcal{D}(\mathcal{A}), \quad \lambda = \omega^2, \tag{283}$$

where the operator matrices $\mathcal{A}$ and $\mathcal{B}$ are defined by (221).

Introduce the operator

$$B_C := C^{-1/2} B_\sigma C^{-1/2} \tag{284}$$

on the natural set

$$\mathcal{D}(B_C) := \{\widetilde{\zeta} \in L_{2,\Gamma} :\ \zeta \in \mathcal{D}(C^{-1/2}) = H_\Gamma^{1/2},\ C^{-1/2}\widetilde{\zeta} \in \mathcal{D}(B_\sigma),\ B_\sigma C^{-1/2}\widetilde{\zeta} \in \mathcal{D}(C^{-1/2})\}. \tag{285}$$

Lemma 9. *The operator B_C has a discrete real spectrum $\{\lambda_k(B_C)\}_{k=1}^{\infty}$ with limit point $+\infty$. The eigenvalues $\{\lambda_k(B_C)\}_{k=1}^{\infty}$ have the same properties as eigenvalues of the operator B_σ (see* (281)*):*

$$-\infty < \lambda_1(B_C) \leqslant \ldots \leqslant \lambda_\varkappa(B_C) < 0 = \lambda_{\varkappa+1}(B_C) = \ldots =$$

$$= \lambda_{\varkappa+q}(B_C) < \lambda_{\varkappa+q+1}(B_C) \leqslant \ldots \leqslant \lambda_k(B_C) \leqslant \ldots. \quad (286)$$

Proof. Consider the eigenvalue problem

$$B_C\xi = C^{-1/2}B_\sigma C^{-1/2}\xi = \lambda\xi. \quad (287)$$

If $\xi \in \mathcal{D}(B_C)$ then $\xi \in \mathcal{D}(C^{-1/2})$ and

$$B_\sigma\widetilde{\xi} = \lambda C\widetilde{\xi}, \quad \widetilde{\xi} = C^{-1/2}\xi \in \mathcal{D}(B_\sigma). \quad (288)$$

Conversely, if $\widetilde{\xi}$ is a solution to equation (288) then $B_\sigma\widetilde{\xi} = B_\sigma C^{-1/2}\xi = \lambda C^{1/2}\xi \in \mathcal{D}(C^{-1/2})$ and equation (287) holds.

If $\lambda = 0$ in problem (288) then $\widetilde{\xi} \in \mathrm{Ker}B_\sigma \neq \{0\}$ and therefore $\lambda = 0$ is a q – multiple eigenvalue of the operator B_C. Introduce the resolution

$$L_{2,\Gamma} = \widehat{L}_{2,\Gamma} \oplus E_q, \quad E_q := \mathrm{Ker}B, \quad \mathrm{dim}E_q = q < \infty, \quad (289)$$

and use the fact that in this resolution problem (288) has the form

$$\widehat{B}_\sigma\widehat{\xi} = \lambda(C\widehat{\xi} + C\xi_q), \quad (290)$$

$$\widehat{\xi} := \widehat{P}\widetilde{\xi} = \widehat{P}\widehat{\xi} \in \widehat{L}_{2,\Gamma}, \quad \xi_q = P_q\widetilde{\xi} = P_q\xi_q \in E_q, \quad (291)$$

where $\widehat{P}$ and P_q are orthoprojections on the subspaces (289).

If we will act from the left in (290) by the operators $\widehat{P}$ and P_q we will have the following system of equations

$$\widehat{B}_\sigma\widehat{\xi} = \lambda(\widehat{P}C\widehat{P}\widehat{\xi} + \widehat{P}CP_q\xi_q), \quad (292)$$

$$0 = \lambda(P_qC\widehat{P}\widehat{\xi} + P_qCP_q\xi_q). \quad (293)$$

Since $\lambda \neq 0$ and P_qCP_q is a q – dimentional positive operator ($q \times q$ – matrix) then from (293) one can find

$$\xi_q = -(P_qCP_q)^{-1}(P_qC\widehat{P})\widehat{\xi}, \quad (294)$$

and therefore (292) takes the form

$$\widehat{B}_\sigma\widehat{\xi} = \lambda\widehat{C}\widehat{\xi}, \quad \widehat{C} := \widehat{P}C\widehat{P} - (\widehat{P}CP_q)(P_qCP_q)^{-1}(P_qC\widehat{P}). \quad (295)$$

Here the operator $\widehat{B}_\sigma$ has a trivial kernel, $\mathrm{Ker}\widehat{B}_\sigma = \{0\}$, and $\widehat{C}$ is a compact and positive (self – adjoint) operator. (Proof of the last properties see in [9], p. 47 – 48.) Further, the operator $\widehat{B}_\sigma$ has a discrete spectrum

$$\sigma(\widehat{B}_\sigma) = \{\lambda_k(B_\sigma)\}_{k=1}^{\varkappa} \cup \{\lambda_k(B_\sigma)\}_{k=\varkappa+q+1}^{\infty}, \quad (296)$$

where $\{\lambda_k(B_\sigma)\}$ are eigenvalues (281).

Represent $\widehat{L}_{2,\Gamma}$ as an orthogonal sum

$$\widehat{L}_{2,\Gamma} = E_{\varkappa} \oplus \check{L}_{2,\Gamma}, \tag{297}$$

where $E_{\varkappa}$ is a $\varkappa$ – dimentional subspace with an orthogonal basis $\{u_k(B_\sigma)\}_{k=1}^{\varkappa}$ corresponding to eigenvalues $\{\lambda_k(B_\sigma)\}_{k=1}^{\varkappa}$ and $\check{L}_{2,\Gamma}$ is an orthogonal complement (with the basis $\{u_k(B_\sigma)\}_{k=\varkappa+q+1}^{\infty}$). Then the operator $\widehat{B}_\sigma$ has the form

$$\widehat{B}_\sigma = \left|\widehat{B}_\sigma\right|^{1/2} J_{\varkappa} \left|\widehat{B}_\sigma\right|^{1/2}, \quad \left|\widehat{B}_\sigma\right| := \left((\widehat{B}_\sigma)^2\right)^{1/2}, \tag{298}$$

$$J_{\varkappa} = \operatorname{diag}(-I_{\varkappa}; \check{I}) = J_{\varkappa}^{-1} = J_{\varkappa}^{*}. \tag{299}$$

It follows from above that $\left|\widehat{B}_\sigma\right| \gg 0$ and therefore there exist bounded and positive operators $\left|\widehat{B}_\sigma\right|^{-1}$, $\left|\widehat{B}_\sigma\right|^{-1/2}$.

Thus, problem (295) takes the form

$$\left|\widehat{B}_\sigma\right|^{1/2} J_{\varkappa} \left|\widehat{B}_\sigma\right|^{1/2} \widehat{\xi} = \lambda \widehat{C} \widehat{\xi}, \tag{300}$$

and after substitution

$$\left|\widehat{B}_\sigma\right|^{1/2} \widehat{\xi} = \eta \tag{301}$$

one can receive the equation

$$J_{\varkappa} \left(\left|\widehat{B}_\sigma\right|^{-1/2} \widehat{C} \left|\widehat{B}_\sigma\right|^{-1/2} \right) \eta = \mu \eta, \quad \mu = \lambda^{-1}. \tag{302}$$

It is evident that here the operator $J_{\varkappa} \left(\left|\widehat{B}_\sigma\right|^{-1/2} \widehat{C} \left|\widehat{B}_\sigma\right|^{-1/2} \right)$ is a $J_{\varkappa}$ – positive compact operator, i.e., it is self – adjoint and positive in the indefinite scalar product

$$[\eta, \zeta] := (J\eta, \zeta)_0 \,. \tag{303}$$

In other words, problem (302) is a spectral problem in the Pontriagin space $\Pi_{\varkappa}$ for compact and positive operator. Therefore, by Theorem from [28], see also [29], [30], problem (302) has exactly $\varkappa$ negative eigenvalues (with account of multiplicities). Another eigenvalues $\{\mu_k\}_{k=\varkappa+1}^{\infty}$ of problem (302) are positive with limit point at 0.

These considerations prove the Lemma, i.e., properties (286). $\square$

On the base of Lemma 9 we come back to problem (283) under assumptions (281).

Theorem 13. *If inequalities* (281) *are fulfilled then problem* (283) *has exactly $\varkappa$ negative eigenvalues (with account of multiplicities) and exactly q zero – eigenvalues. The other eigenvalues of problem* (283) *are positive and have limit point at infinity.*

Proof. It is the same as proof of Lemma 9. Namely, we consider problem (283) with

$$\mathcal{A} = \operatorname{diag}(c^2 \rho_2 A; B_C) \tag{304}$$

and the operator $\mathcal{B}$ from (221) which is bounded and positive definite (see Lemma 8). Since $A \gg 0$ and $A^{-1} \in \mathfrak{S}_\infty$ then the operator $\mathcal{A}$ has a discrete spectrum

$$\sigma(A) = \{c^2\rho_2\lambda_k(A)\}_{k=1}^{\infty} \cup \{\lambda_k(B_C)\}_{k=1}^{\infty}, \tag{305}$$

where $\lambda_k(B_C)$ have properties (286). Therefore one can repeat proof of Lemma 9 not to equation (288) but to equation (283). It proves the theorem. $\square$

As a corollary of Theorem 13 we have the following resulting assertion.

Theorem 14. *(inverse of Lagrange Theorem on Stability).*

If the operator B_σ of potential energy of the system „fluid – gas" is not statically stable in linear approximation, i.e., condition (69) *is not fulfilled and B_σ has properties* (281) *with $\varkappa \geqslant 1$ and $q \geqslant 0$, then problem* (283) *has at least one negative eigenvalue $\lambda = \omega^2 < 0$. Therefore there exists solution $y(t)$ to homogeneous problem* (218) *such that*

$$y(t) = y \exp(t\sqrt{|\,\lambda\,|}), \quad y \in \mathcal{D}(\mathcal{A}), \tag{306}$$

i.e., this solution exponentially increases in time. $\square$

6.6. **The case of motions of the system „heavy fluid – gas".** The considered above problem contains as a special case the problem on small motions of a system „heavy fluid – gas" when surface tension does not taken into account. This last problem is investigated in [21]. Here we mention briefly corresponding results for the case.

First of all, if surface forces do not act and we must take into account only gravity then a free surface of a fluid is horizontal at equilibrium state, i.e., it is perpendicular to direction of gravity action.

Considering small oscillations of the system we must put $\sigma = 0$ in problem (8) – (15). In this, we have $\vec{n} = \vec{e}_3$ on Γ, $\mathcal{L}_\sigma\zeta$ must be changed by $\mathcal{L}_0\zeta := (\rho_1 - \rho_2)g\zeta$, because $\cos(\vec{n}, \vec{e}_3) = 1$. Besides, condition (14) must be omitted. Therefore the operator $B_\sigma\,|_{\sigma=0} =: B_0$ of potential energy has the form (see (51))

$$B_0 = (\rho_1 - \rho_2)gI, \quad \mathcal{D}(B_0) = L_{2,\Gamma}. \tag{307}$$

Since the operator B_0 is positive definite ($\rho_1 - \rho_2 > 0$, $g > 0$) then the system „heavy fluid – gas" is statically stable.

In spectral problem (3) – (68) we now must change B_σ by B_0, and the functionals (70) and (123) have the forms

$$F_1(\Phi_1; \Phi_2) = \frac{\displaystyle\sum_{k=1}^{2} \rho_k \int\limits_{\Omega_k} |\nabla\Phi_k|^2 \, d\Omega_k}{\displaystyle\rho_2 c^{-2} \int\limits_{\Omega_2} |\Phi_2|^2 \, d\Omega_2 + ((\rho_1 - \rho_2)g)^{-1} \|P_\Gamma(\rho_1\Phi_1 - \rho_2\Phi_2)\|_0^2}, \tag{308}$$

$$F_2(\Phi_1;\Phi_2) = \frac{c^2\rho_2 \int\limits_{\Omega_2} |\Delta\Phi_2|^2 \, d\Omega_2 + (\rho_1-\rho_2)g \int\limits_{\Gamma} |\zeta|^2 \, d\Gamma}{\sum\limits_{k=1}^{2} \rho_k \int\limits_{\Omega_k} |\nabla\Phi_k|^2 \, d\Omega_k}, \tag{309}$$

and conditions (124) must be taken into account. The main spectral problem (107) now has the some form with substitution $(g(\rho_1-\rho_2))^{-1/2}$ instead of $B_\sigma^{-1/2}$.

For the case $\sigma = 0$ Theorems 1, 2, 3, 4, 5, 6, 7 are valid also (with corresponding modifications). As in Subsection 5.4, we have here acoustic and surface waves, but now the asymptotic behavior of the eigenvalues of surface waves has another form.

In problem on strong solvability of an initial boundary value problems (Section 6) we come again to Cauchy problem (218) for hyperbolic equation in Hilbert space $\mathcal{H}(\Omega) = L_{2,\Omega_2} \oplus L_{2,\Gamma}$, but now the operator matrix $\mathcal{A}$ has not form (221) but a new form

$$\mathcal{A} := \operatorname{diag}(c^2\rho_2 A; (g(\rho_1-\rho_2))^{-1} C^{-1}) \tag{310}$$

with

$$\mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \oplus \mathcal{D}(C^{-1}). \tag{311}$$

It is evident that the operator $\mathcal{A}$ is positive definite and self-adjoint on domain $\mathcal{D}(\mathcal{A})$. Therefore Theorems 9 – 12 with new assumptions,

$$\zeta^0 \in \mathcal{D}(C^{-1/2}) = H_\Gamma^{1/2}, \quad \zeta^1 \in L_{2,\Gamma}, \tag{312}$$

and with corresponding simplified assertions hold. For instance, in Theorem 11 we have instead of (266), (267):

$$\zeta = (\vec{w}_1 \cdot \vec{n})_\Gamma = (\vec{w}_2 \cdot \vec{n})_\Gamma \in C(\mathbb{R}_+; H_\Gamma^{1/2}),$$

$$(p_1 - p_2)_\Gamma = P_\Gamma(\rho_1\Phi_1 - \rho_2\Phi_2)_\Gamma = g(\rho_1-\rho_2)\zeta \in C(\mathbb{R}_+; H_\Gamma^{1/2}).$$

At last, new considered system „heavy fluid – gas" is dynamical stable, it has a discrete positive spectrum $\{\lambda_k\}_{k=1}^\infty$, i.e., $\lambda_k = \omega_k^2$, where $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_k \leqslant \ldots$, $\lambda_k \longrightarrow +\infty$ $(k \longrightarrow \infty)$. It means that all frequencies of oscillations are real.

Remark in conclusion that on the base of problem considered in the paper the authors plan to investigate correspondent problems on small oscillations for rotating system consisting of ideal fluid and a gas, viscous fluid and gas, and all the same problems for nonlinear case.

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Список литературы

[1] Lamb, H. Fluid Dynamics. Moscow, Leningrad: Gostechizdat, 1947, pp. 928 (in Russian).

[2] Moiseev, N.N. and Rumyantsev, V.V. Dynamics of a Body with Cavities Containing Fluid. Moscow: Nauka, 1965, pp. 440 (in Russian).

[3] Mikishev, G.H. and Rabinovich, B.I. Dynamics of a Rigid Body with cavities Partially Filled with Fluid. Moscow: Mashinostroenie, 1968, pp. 532 (in Russian).

[4] Lukovsky, I.A., Barnyak, M.Ya, and Komarenko A.N. Priblizhennije Metody Reshenija Zadach Dynamiki Ogranichennogo Objoma Zhidkosti. Kiev: Naukova Dumka, 1984, pp. 232 (in Russian).

[5] Babskii, V.G., Kopachevsky, N.D., Myshkis, A.D., Slobozhanin, L.A, and Tyuptsov, A.D.. Gidromehanika Nevesomosty. Moscow: Nauka, 1976, pp. 504 (in Russian).

[6] Myshkis, A.D., Babskii, V.G., Kopachevsky, N.D., Slobozhanin, L.A., and Tyuptsov, A.D. Low – Gravity Fluid Mechanics. Springer Verlag, Heidelberg, 1987, pp. 583.

[7] Myshkis, A.D., Babskii, V.G., Zhukov, M.Y., Kopachevsky, N.D., Slobozhanin, L.A., and Tyuptsov, A.D. Methods of Solving Problems of Fluid mechanics for conditions of Low – Gravity. Kiev: Naukova Dumka, 1992, pp. 592 (in Russian).

[8] Kopachevsky, N.D., Krein S.G., and Ngo Zui Kan. Operator Methods in Linear Fluid Dynamics: Evolution and Spectral Problems, Moscow: Nauka, 1989, pp. 416 (in Russian).

[9] Kopachevsky, N.D., and Krein, S.G. Operator Approach to Linear Problems of Hydrodynamics. Volume 1: Self – adjoint Problems for an Ideal Fluid. Birkhauser Verlag, Basel, 2001, pp. 384. (Operator theory: Advances and Applications, Vol. 128.)

[10] Kopachevsky, N.D., and Krein, S.G. Operator Approach to Linear Problems of Hydrodynamics. Volume 2: Nonself – adjoint Problems for Viscous Fluid. Birkhauser Verlag, Basel, 2003, pp. 444. (Operator theory: Advances and Applications, Vol. 146.)

[11] Kopachevsky, N.D., and Vronsky, B.M. Samosopriazhennije operatornije puchki, porozhdennije odnoy zadachey o kolebanijah systemy „zhidkost – gas" // Materialy 19 shkoly po teorii operatorov v functsional'nih prostranstvah. – Ternopol (Ukraine), 1984, p. 25 – 26 (in Russian).

[12] Kopachevsky, N.D., Temchenko, T.P., and Vronsky, B.M. Kolebanija chastichno dissipativnich gidrosistem // Materialy 12 shkoly po teorii operatorov v functsional'nih prostranstvah. – Tambov (Ukraine), 1987, p. 40 (in Russian).

[13] Kopachevsky, N.D., and Vronsky, B.M. Initial Boundary Value and Spectral Problems on Small Oscillations of the system „ideal fluid – gas" // Spectral and Evolution Problems: Proceedings of the 4 -th Crimea Autumn Math. School – Sympos.. Vol 4. – Simferopol (Ukraine): 1995, – pp. 107 – 111.

[14] Vronsky, B.M. Oscillations of a hydrosystem „fluid – gas" in a bounded domain // Mark Krein Intern. Conference. Operator Theory and Appl. August 18-22, 1997, Odessa (Ukraine). Book of abstracts, p. 127 – 128.

[15] Vronsky, B.M. On localization of he Spectrum of Some Hydrodynamic Problem // Spectral and Evolution problems: Proceedings of the 8 - th Crimea Autumn Math. School – Symp. Vol. 10. – Simferopol (Ukraine), 2000, pp. 106 – 109.

[16] Vronsky, B.M. On structure of the Spectrum of Some Hydrodynamic Problem // Spectral and Evolution problems: Proceedings of the 8 - th Crimea Autumn Math. School – Symp. Vol. 10. – Simferopol (Ukraine), 2000, pp. 110 – 111.

[17] Lukovsky, I.A., and Timoha, A.N. Ob akusticheskom vozdeystvii na svobodnuju poverhnost' ogranichennogo ob'ema zhidkosti // Acoustic journ., 1991, V. 37, № 1, p. 144 – 149 (in Russian).

[18] Lukovsky, I.A., and Timoha, A.N. Prostranstvennije dvizhenija rezervuara s zhidkost'ju, nahodiascheysia pod deystviem vibroakusticheskoy nagruzki // Kiev (Ukraine), 1990, pp. 57 (preprint Instituta matematiki AN Ykrainskoy SSR, № 90 (33) (in Russian).

[19] Lukovsky, I.A., and Timoha, A.N. Sobstvennije kolebanija svobodnoy poverhnosti ogranichennogo ob'ema zhidkosty, vzaimodeystvujuschey s akusticheskim polem // Doklady akademii Nauk USSR, sep. phiz. – mat. i tehn. nauk, 1990, № 12, pp. 24 – 26 (in Russian).

[20] Lukovsky, I.A., and Timoha, A.N. O stabilizatsii poverhnosti razdela zhidkosti i gaza pri vzaimodeystvii s akusticheskimi poliami v gaze // Mehanika zhidkosti i gaza, 1991, № 3, pp. 80 – 86 (in Russian).

[21] Vronsky, B.M. Initial boundary value and spectral problems generated by the problem on small motions of the system „fluid – gas". PhD – thesis, Ukraine, Simferopol, Taurida National V.Vernadsky University. Math. Dep., 2007, 115pp.

[22] Landau, L.D., and Lifshitz, E.M. Mehanika sploshnih sred. Moscow, GITTL, 1953, pp. 788 (in Russian).

[23] Mak-Konnel, A.Dzh. Vvedenije v tenzorniy analiz s prilozhenijami k geometrii, mehanike i fizike. Moscow: Fizmatgiz, 1963 (in Russian).

[24] Blyashke, B. Differential'naya geometriya i geometricheskije osnovy teorii otnositel'nosti Ayshteyna. Moscow: ONTI, 1935 (in Russian).

[25] Mikhlin, S.G. Variatsionnije metody v matematicheskoy phizike. Moscow, Nauka, 1970, pp. 310.

[26] Berezansky, Yu.M. Razlozheniye po sobstvennim funktsijam samosopriazhennih operatorov. Kiev: Naukova Dumka, 1965, pp. 800 (in Russian).

[27] Gagliardo, E. Caratterizazioni delle trace sullo frontiera relative ad alcune classi de funzioni in „n" variabili // Rend. Sem. mat. Univ. Padova, 1957, V. 27, pp. 284 – 305.

[28] Pontryagin, L.S. Hermitian operators in spaces with indefinite metrics // Izv. AN SSSR, Series Math., 8(6), 1944, pp. 243 – 280 (in Russian).

[29] Azizov, T.Ya., and Iohvidov, I.S. Foundations of the Theory of Linear Operators in Spaces with Indefinite Metrics. Moscow: Nauka, 1986, pp. 352 (in Russian).

[30] Iohvidov, I.S., Krein, M.G., and Langer, H. Introduction to the Spectral Theory of Operators in Spaces with an indefinite Metric. Berlin: Akademic Verlag, 1982, Math. Research, Band 9, pp. 120.